

Revised formulas for diffraction effects with point and extended sources

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Revised formulas to estimate diffraction effects in radiometry for point and extended sources are derived. They are found to work as well as or better than previous formulas. In some instances the formulas can be written in closed form; otherwise their evaluation entails performing simple integrations as indicated. Formulas have been found for nonlimiting apertures, large defining apertures, and pinhole apertures. Examples of all three types of application are presented.

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1. Introduction

The wave nature of light restricts the validity of geometrical optics. However, practical radiometry often relies on geometrical optics to relate source radiance, geometrical throughput of a setup, and detector efficiency, so that one of these quantities can be found if the other two are known. Consequently, the limitations of geometrical optics introduce diffraction errors into measurements. Compensating corrections, which are based on the theory of diffraction at apertures, can be an integral part of measurements. For a given setup, diffraction errors tend to increase with wavelength λ , so corrections are a critical part of radiometry in the long-wavelength infrared region.

Whereas a real optical setup can be quite complicated, one can often estimate diffraction errors by considering one of the following two hypothetical situations: Either a defining aperture stands between an (extended) source and a detector or a nonlimiting aperture (i.e., a baffle) is between those two optics. In either case the geometry of a hypothetical setup should mimic a portion of the real setup that contributes to diffraction errors, so diffraction affects flux incident upon the hypothetical detector in the same way that diffraction influences real measurements. Such a defining aperture functions in one of two ways. Either the detector is underfilled, so both the fully and the partially illuminated regions of the detecting plane are contained wholly within the detec-

tor, or the reciprocal situation is true, so the source overfills completely all regions of the source plane seen through the aperture by any part of the detector. Also, if one has several baffles situated in series between the same source and detector, the effects of each baffle may contribute to the total diffraction error in approximately additive fashion, permitting the baffles to be considered one at a time.

Determination of diffraction effects in the above hypothetical situations has been of long-standing interest in radiometry. Considerable effort has been devoted to the study of setups that have cylindrical symmetry, to which the present study is limited. Blevin¹ analyzed the loss or absence of geometrically expected flux caused by a limiting aperture, using the mathematical formulation of Focke.² Steel *et al.*³ addressed this problem as well, dealing explicitly with effects of using an extended source in the limiting case. Those researchers also discussed the nonlimiting case. Boivin presented a more-detailed examination of the nonlimiting case, predicting excess flux because of diffraction in most practical situations.⁴ He used detailed formulas for the flux pattern derived originally by Lommel.⁵ Boivin also considered the problem of a nonlimiting aperture and an extended source.⁶

It is worthwhile to reexamine such issues, especially because common formulas used to estimate diffraction errors work to disparately different degrees in various situations. Revised formulas that estimate errors more reliably would be particularly helpful. Conversely, a detailed numerical evaluation of the diffraction errors, at least within the Kirchhoff formulation, is now often practical, and the present author has developed several programs to do just this. Assessment of the reliability of such programs

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would be enhanced by a more-detailed means of comparing numerical and analytical formulas for diffraction errors, an objective for this author. Furthermore, such formulas play a special role in practice because they are often accurate when numerical work is impractical, e.g., for small λ .

In this study, therefore, revised formulas are derived for diffraction errors; these formulas have worked as well as, or better than, the formulas cited above in all cases examined. But these formulas also have a limited range of applicability, and the pursuit of formulas that are robust in a wider range of applications seems difficult. For the case of a point source the formulas are relatively complete, and their generalization to cases of extended sources is also derived. Intrinsic inaccuracies of the formulas are not given, because several of the approximations used in their derivation can each prove important, depending on the situation. (So a formula's validity is best assessed in practice.)

Establishing the validity of the Fresnel and paraxial approximations is a prerequisite to this study. Given these prerequisites, a preliminary expression for spectral power incident upon the detector is derived in Section 2. This expression depends on the source spectral radiance, the geometry considered, and Lommel's results; although it is an integral expression, its evaluation entails only a single (i.e., one-dimensional) integration. The relevant integrand can be related to the spectral power incident upon a detector in the case of a point source. Helpful approximations for this integrand are derived in Section 3. In Section 4 the results of Sections 2 and 3 are combined to produce explicit (already integrated) expressions for the spectral power in cases of extended sources. Each of these geometry-dependent expressions can be written in closed form while validity is maintained over a wide range of circumstances. Otherwise, the results of Section 3 vastly simplify the integration indicated in Section 2, so diffraction errors can be estimated easily in all cases. The results are demonstrated in Section 5 for several types of situation. The presentation of conclusions and a technical appendix form the balance of this paper.

2. Fresnel Diffraction by a Circular Aperture for an Extended Source

A source-aperture-detector configuration, for which diffraction effects are considered here, is specified by

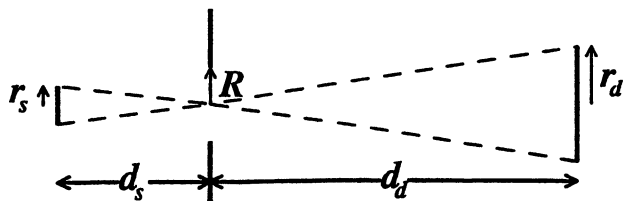


Fig. 1. Schematic diagram of the source-aperture-detector setup with relevant dimensions labeled. Angles subtended by the source and the detector are indicated. The particular setup shown features a nonlimiting aperture.

source radius r_s , aperture radius R , source-aperture distance d_s , detector radius r_d , and aperture-detector distance d_d (Fig. 1). Given the spectral radiance of the source, $L_\lambda(\lambda)$, the spectral radiant flux incident upon the detector, $\Phi_\lambda(\lambda)$, can in principle be found. It is helpful to introduce three parameters, which are given by

$$u = \frac{2\pi}{\lambda} R^2 \left(\frac{1}{d_s} + \frac{1}{d_d} \right), \quad v_s = \frac{2\pi}{\lambda} \frac{R r_s}{d_s}, \quad v_d = \frac{2\pi}{\lambda} \frac{R r_d}{d_d}. \quad (1)$$

The angles subtended at the aperture by the source and the detector, which have the ratio $(2r_s/d_s)/(2r_d/d_d) = v_s/v_d$, will generally not be equal, and it is convenient to define two new variables, which are given by

$$v_0 = \text{Max}(v_s, v_d), \quad \sigma = \frac{\text{Min}(v_s, v_d)}{\text{Max}(v_s, v_d)}. \quad (2)$$

Of the angle subtended by the source and the angle subtended by the detector, the larger is

$$\Psi = \frac{\lambda v_0}{\pi R}. \quad (3)$$

The condition of having a defining aperture, such that the geometrical throughput of the configuration is specified by the aperture in combination with either the source or the detector, is expressed by the relation

$$u < (1 - \sigma)v_0 = |v_s - v_d|. \quad (4)$$

The condition of having a nonlimiting aperture is expressed by the relation

$$u > (1 + \sigma)v_0 = v_s + v_d. \quad (5)$$

Henceforth it is assumed that one or the other condition is true.

For convenience, the center of the aperture is defined as the origin and the z axis is chosen as the optical axis. For two points, $\mathbf{r}_s = (x_s, y_s, -d_s)$ on the source and $\mathbf{r}_d = (x_d, y_d, d_d)$ on the detector, the line segment connecting these points intersects the x - y plane at a point $\mathbf{r}_M = (x_M, y_M, 0)$, which implicitly depends on \mathbf{r}_s and \mathbf{r}_d . Define $f(\mathbf{r}_s, \mathbf{r}_d)$ through

$$f(\mathbf{r}_s, \mathbf{r}_d) = \frac{\sqrt{x_M^2 + y_M^2}}{R}. \quad (6)$$

Using Lommel's solution for the Fresnel diffraction in the paraxial approximation by a circular aperture, we can write

$$\Phi_\lambda(\lambda) = \frac{L_\lambda(\lambda)}{(d_s + d_d)^2} \int_{\text{Source}} d^2\mathbf{r}_s \int_{\text{Detector}} d^2\mathbf{r}_d \times |\alpha[u, f(\mathbf{r}_s, \mathbf{r}_d)u]|^2. \quad (7)$$

If we simplify notation by defining $v = f(\mathbf{r}_s, \mathbf{r}_d)u$, $|\alpha(u, v)|^2$ is most conveniently given by

$$|\alpha(u, v)|^2 = \begin{cases} 1 + V_0^2(u, v) + V_1^2(u, v) \\ - 2V_0(u, v)\cos\left(\frac{u + \frac{v^2}{u}}{2}\right) \\ - 2V_1(u, v)\sin\left(\frac{u + \frac{v^2}{u}}{2}\right) & v < u \\ U_1^2(u, v) + U_2^2(u, v) & v > u \end{cases} \quad (8)$$

Lommel functions V and U are given by

$$U_n(u, v) = \sum_{s=0}^{\infty} (-)^s \left(\frac{u}{v}\right)^{n+2s} J_{n+2s}(v),$$

$$V_n(u, v) = \sum_{s=0}^{\infty} (-)^s \left(\frac{v}{u}\right)^{n+2s} J_{n+2s}(v), \quad (9)$$

where the J cylindrical functions are Bessel functions.

Integrations over \mathbf{r}_s and \mathbf{r}_d can best be carried out in cylindrical polar coordinates r_s', r_d', θ_s , and θ_d , where we have $\mathbf{r}_s = (r_s' \cos \theta_s, r_s' \sin \theta_s, -d_s)$, etc. The result is

$$\begin{aligned} \Phi_\lambda(\lambda) &= \frac{L_\lambda(\lambda)}{(d_s + d_d)^2} \int_0^{r_s} dr_s' r_s' \int_0^{2\pi} d\theta_s \int_0^{r_d} dr_d' r_d' \\ &\quad \times \int_0^{2\pi} d\theta_d |\alpha[u, f(\mathbf{r}_s, \mathbf{r}_d)u]|^2 \\ &= \frac{2\pi r_s^2 r_d^2 L_\lambda(\lambda)}{(d_s + d_d)^2 \sigma^2} \int_0^\sigma d\sigma' \sigma' \int_0^1 d\zeta \zeta \\ &\quad \times \int_0^{2\pi} d\theta |\alpha[u, v_0(\sigma'^2 + \zeta^2 - 2\zeta\sigma' \cos \theta)^{1/2}]|^2. \end{aligned} \quad (10)$$

[In the last step, one angular integration can be eliminated because of cylindrical symmetry; integration over the dimensionless $\sigma'(\zeta)$ is a reexpression of the radial integration for the end optic subtending a smaller (larger) angle at the aperture. Rearrangement of the second argument of α is found from geometric considerations.] If we introduce the new function

$$I(u, v, \tau) = \int_0^\tau d\tau' \tau' |\alpha(u, v\tau')|^2 = \frac{1}{v^2} \int_0^{v\tau} dv' v' |\alpha(u, v')|^2, \quad (11)$$

we can derive a more useful expression:

$$\begin{aligned} \Phi_\lambda(\lambda) &= \frac{2\pi r_s^2 r_d^2 L_\lambda(\lambda)}{(d_s + d_d)^2} \int_{-1}^1 dx \frac{I(u, v_0, 1 + \sigma x)}{1 + \sigma x} \\ &\quad \times \{(1 - x^2)[(2 + \sigma x)^2 - \sigma^2]\}^{1/2}, \end{aligned} \quad (12)$$

as is demonstrated in Appendix A.

Thus the evaluation of $\Phi_\lambda(\lambda)$ requires only a single integration of an ostensibly known integrand. This integration can be surprisingly easy and accurate, whether it is carried out numerically or (albeit approximately) analytically. Meanwhile, without diffraction we have

$$\begin{aligned} \Phi_\lambda^{(0)}(\lambda)|_{\text{nonlim.}} &= \frac{\pi^2 r_s^2 r_d^2 L_\lambda(\lambda)}{(d_s + d_d)^2} \\ \Phi_\lambda^{(0)}(\lambda)|_{\text{lim.}} &= \frac{\pi^2 \sigma^2 R^2 \Psi^2 L_\lambda(\lambda)}{4} = \frac{u^2}{v_0^2} \frac{\pi^2 r_s^2 r_d^2 L_\lambda(\lambda)}{(d_s + d_d)^2} \end{aligned} \quad (13)$$

for the nonlimiting and the limiting cases, respectively.

3. Formulas for $I(u, v_0, \tau)$

$I(u, v_0, \tau)$, defined in Eq. (11), is closely related to the total flux incident upon a detector in the case of a point source on the z axis behind an aperture. Below, approximate formulas for $I(u, v_0, \tau)$ are given, and these formulas pertain to the cases of (1) a large nonlimiting aperture (baffle), (2) a large defining aperture, and (3) a pinhole defining aperture. Several approximations for $I(u, v_0, \tau)$ in the nonlimiting case have been analyzed by Boivin.⁴ The formula found here is less compact than those presented by Boivin because it retains more contributions, which the present author has found widens its scope of applicability considerably. Formulas for $I(u, v_0, \tau)$ for large and pinhole defining apertures have been taken from Steel *et al.*³ and from Born and Wolf.⁷

A. Large Nonlimiting Aperture

For a large nonlimiting aperture one has $v_0\tau < u$ for all τ of interest, so the formula $v < u$ for $\alpha(u, v)$ is preferred. When the asymptotic formula for Bessel functions,

$$J_m(v) \sim \sqrt{\frac{2}{\pi v}} \cos\left(v - \frac{m\pi}{2} - \frac{\pi}{4}\right), \quad (14)$$

is used, approximate forms for V_0 and V_1 become, for v/u sufficiently smaller than 1:

$$\begin{aligned} V_0(u, v) &\approx \sqrt{\frac{2}{\pi v}} \frac{\cos(v - \pi/4)}{1 - v^2/u^2}, & 1 \ll v < u, \\ V_1(u, v) &\approx \sqrt{\frac{2}{\pi v}} \frac{v \sin(v - \pi/4)}{u(1 - v^2/u^2)}, & 1 \ll v < u. \end{aligned} \quad (15)$$

Substitution of relations (15) into Eq. (11) permits us to write

$$I(u, v_0, \tau)|_{1 \ll v_0 \tau < u} \approx \frac{1}{v_0^2} \int_0^{v_0 \tau} dv' v' \times \left\{ 1 + \frac{1}{\pi v'} \left[\frac{1 + v'^2/u^2}{(1 - v'^2/u^2)^2} \right] + \frac{1}{\pi v'} \frac{\sin(2v')}{1 - v'^2/u^2} - \sqrt{\frac{2}{\pi v'}} \frac{\cos[(u + v')^2/(2u) - \pi/4]}{(u + v')/u} - \sqrt{\frac{2}{\pi v'}} \frac{\cos[(u - v')^2/(2u) + \pi/4]}{(u - v')/u} \right\}. \quad (16)$$

This integrand is a poor approximation for small v' , but the resultant error is minimal. Of the five terms in braces in the integrand, contributions from the first two can be integrated exactly. Contributions from the third term prove to be small, because of the oscillatory nature of that term, and are neglected. Although the remaining two terms are also oscillatory, the oscillations are not periodic, and the terms are weighted by a lower inverse power of v' than is the second or the third term. In practice, the last two terms prove to be important, but their contributions to the integral can be represented only approximately. Expressions for their contributions have been arrived at here through integration by parts, during which the trigonometric functions were one of the two factors in the integrand. Hence the terms being neglected are of higher inverse powers of u or v_0 . Thus $I(u, v_0, \tau)$ is approximated by

$$I(u, v_0, \tau)|_{1 \ll v_0 \tau < u} \approx \frac{\tau^2}{2} + \frac{u^2 \tau}{\pi v_0 (u^2 - v_0^2 \tau^2)} - \frac{u^2}{v_0^2} \sqrt{\frac{2v_0 \tau}{\pi}} \frac{\sin[(u + v_0 \tau)^2/(2u) - \pi/4]}{(u + v_0 \tau)^2} + \frac{u^3}{v_0^2} \sqrt{\frac{2v_0 \tau}{\pi}} \cos\left[\frac{(u + v_0 \tau)^2}{2u} - \frac{\pi}{4}\right] \times \left[+ \frac{2}{(u + v_0 \tau)^4} - \frac{1}{2v_0 \tau (u + v_0 \tau)^3} \right] + \frac{u^2}{v_0^2} \sqrt{\frac{2v_0 \tau}{\pi}} \frac{\sin[(u - v_0 \tau)^2/(2u) + \pi/4]}{(u - v_0 \tau)^2} + \frac{u^3}{v_0^2} \sqrt{\frac{2v_0 \tau}{\pi}} \times \cos\left[\frac{(u - v_0 \tau)^2}{2u} + \frac{\pi}{4}\right] \left[- \frac{2}{(u - v_0 \tau)^4} - \frac{1}{2v_0 \tau (u - v_0 \tau)^3} \right], \quad (17)$$

an expression that, besides having the shortcomings already mentioned, neglects integration constants related to the $v' = 0$ limit of integration. The results presented below suggest that such contributions are small.

B. Large Defining Aperture

For a large defining aperture one requires $I(u, v_0, \tau)$ only for $v_0 \tau > u$, and it is convenient to write

$$I(u, v_0, \tau)|_{1 \ll u < v_0 \tau} = \frac{1}{v_0^2} \left[\int_0^\infty dv' v' |\alpha(u, v')|^2 - \int_{v_0 \tau}^\infty dv' v' |\alpha(u, v')|^2 \right] \approx \frac{1}{v_0^2} \left[\frac{u^2}{2} - \int_{v_0 \tau}^\infty dv' v' |\alpha(u, v')|^2 \right]. \quad (18)$$

This approximation replaces the actual diffraction loss with the difference between such a loss and the analogous loss for an arbitrarily large v_0 . Because the latter loss is much smaller, this replacement is acceptable.

Again using the asymptotic form of Bessel functions, we find that

$$U_1(u, v) \approx \sqrt{\frac{2}{\pi v}} \frac{u \sin(v - \pi/4)}{1 - u^2/v^2},$$

$$U_2(u, v) \approx -\sqrt{\frac{2}{\pi v}} \left(\frac{u}{v} \right)^2 \frac{\cos(v - \pi/4)}{1 - u^2/v^2}, \quad (19)$$

which leads to the approximation cited by Steel *et al.*³ Their approximation was perhaps misprinted and appears here as:

$$I(u, v_0, \tau)|_{1 \ll u < v_0 \tau} \approx \frac{u^2}{2v_0^2} \left[1 - \frac{2v_0 \tau}{\pi(v_0^2 \tau^2 - u^2)} + \frac{\cos(2v_0 \tau)}{\pi(v_0^2 \tau^2 - u^2)} + \dots \right]. \quad (20)$$

C. Pinhole Defining Aperture

The Fraunhofer result for a pinhole aperture is well known and can be found, for instance, in Born and Wolf.⁷ In the present notation this result is

written as

$$I(u, v_0, \tau)|_{\text{pinhole}} \approx \frac{u^2}{2v_0^2} [1 - J_0^2(v_0\tau) - J_1^2(v_0\tau)]. \quad (21)$$

Simple approximations to the bracketed term can yield a more convenient form for relation (21). For instance, a relative error of less than 0.01 is achieved with the substitution, for an arbitrary argument x ,

$$1 - J_0^2(x) - J_1^2(x) \approx \begin{cases} \frac{x^2}{4} - \frac{x^4}{32} + \frac{5x^6}{2304} - \frac{7x^8}{73728} + \frac{7x^{10}}{2457600} & x < 2.67 \\ 1 - \frac{2}{\pi x} + \frac{\cos(2x)}{\pi x^2} & x \geq 2.67 \end{cases} \quad (22)$$

More-accurate but more-complicated approximations are also available.

4. Expressions for Diffraction Errors

It is convenient to specify a diffraction error in terms of the ratio of the actual flux incident upon the detector to the flux that would be incident without diffraction. This ratio is F_1 for defining apertures and F_2 for nonlimiting apertures. Correcting a measurement for diffraction involves division of the measured flux by F_1 or F_2 , which are

$$F_1(u, v_0, \sigma) = \frac{1}{\pi} \int_{-1}^1 dx \frac{\{(1-x^2)[(2+\sigma x)^2 - \sigma^2]\}^{1/2}}{1+\sigma x} \times \left[\frac{2v_0^2 I(u, v_0, 1+\sigma x)}{u^2} \right], \quad (23)$$

$$F_2(u, v_0, \sigma) = \frac{1}{\pi} \int_{-1}^1 dx \frac{\{(1-x^2)[(2+\sigma x)^2 - \sigma^2]\}^{1/2}}{1+\sigma x} \times [2I(u, v_0, 1+\sigma x)]. \quad (24)$$

If suitably accurate explicit expressions for such integrals are not available, Gauss-Chebyshev quadrature permits efficient numerical integration based on this formula for smooth $f(x)$:

$$\frac{1}{\pi} \int_{-1}^1 dx \sqrt{1-x^2} f(x) \approx \frac{1}{N} \sum_{k=1}^N f \left[\cos \left(\pi \frac{2k-1}{2N} \right) \right] \times \sin^2 \left(\pi \frac{2k-1}{2N} \right). \quad (25)$$

The author has found that, if such quadrature is not well converged for $N \approx 12$, approximate analytical expressions estimate the integrals reasonably, and these expressions are derived below. That is, recommendations for how to combine Eqs. (23) and (24) and the results of Section 3 are given for estimating diffraction effects in cases of extended sources, whereas F_1 and F_2 are simply the last quantity in brackets in each of Eqs. (23) and (24) for point sources located on the optic axis.

A. Large Nonlimiting Aperture

We can reexpress the results of Eq. (17) in the abbreviated form for the integrand in Eq. (24) through the definition

$$\begin{aligned} & \frac{\{(1-x^2)[(2+\sigma x)^2 - \sigma^2]\}^{1/2}}{1+\sigma x} 2I(u, v_0, 1+\sigma x) \\ &= \{(1-x^2)[(2+\sigma x)^2 - \sigma^2]\}^{1/2} (1+\sigma x) \\ &+ \frac{2u^2 \{(1-x^2)[(2+\sigma x)^2 - \sigma^2]\}^{1/2}}{\pi v_0 (u^2 - v_0^2 (1+\sigma x)^2)} \\ &+ \sqrt{1-x^2} [A_1(x) \sin \theta_+ + A_2(x) \cos \theta_+ \\ &+ A_3(x) \sin \theta_- + A_4(x) \cos \theta_-]. \end{aligned} \quad (26)$$

Here we have

$$\theta_{\pm} = \left(\frac{u \pm v_0}{\sqrt{2u}} \pm \frac{v_0 \sigma}{\sqrt{2u}} x \right)^2 \mp \frac{\pi}{4}, \quad (27)$$

and $A_1(x)$ – $A_4(x)$ are given by

$$A_1(x) = -\frac{2u^2}{v_0^2} \left[\frac{2v_0(1+\sigma x)}{\pi} \right]^{1/2} \times \frac{[(2+\sigma x)^2 - \sigma^2]^{1/2}}{(1+\sigma x)[u + v_0(1+\sigma x)]^2} \quad (28)$$

and so forth. It is helpful to introduce the six parameters

$$\begin{aligned} a' &= -\frac{v_0 \sigma}{u + v_0}, & b' &= \frac{v_0 \sigma}{u - v_0}, \\ \alpha &= \frac{1 - \sqrt{1 - a'^2}}{a'}, & \beta &= \frac{1 - \sqrt{1 - b'^2}}{b'}, \\ s &= \frac{\sigma}{4 - \sigma^2}, & \Sigma &= \frac{-1 + \sqrt{1 - 4s^2}}{2s}, \end{aligned} \quad (29)$$

as well as the functional

$$I_2[A, a, b, c] = \frac{1}{\pi} \int_{-1}^1 dx \sqrt{1-x^2} \times \exp[i\{(a+bx)^2 + c\}] A(x) \quad (30)$$

and the function

$$I_3(u, v_0, \sigma) = \frac{1}{\pi} \int_{-1}^1 dx \frac{\{(1-x^2)[(2+\sigma x)^2 - \sigma^2]\}^{1/2}}{u^2 - v_0^2 (1+\sigma x)^2}. \quad (31)$$

In Appendix A approximate formulas for I_2 and I_3 are derived for arguments of interest. For $a > |b| > 0$ and $|ab| \gg 1$ we have

$$I_2[A, a, b, c] \approx \frac{\exp[i(a^2 + b^2 + c)]}{4\sqrt{\pi}|ab|^{3/2}} \times \left\{ \sin\left(2|ab| - \frac{\pi}{4}\right) \left[\left(\frac{a}{a+b}\right)^{3/2} A(1) + \left(\frac{a}{a-b}\right)^{3/2} A(-1) \right] - i \frac{ab}{|ab|} \cos\left(2|ab| - \frac{\pi}{4}\right) \left[\left(\frac{a}{a+b}\right)^{3/2} A(1) - \left(\frac{a}{a-b}\right)^{3/2} A(-1) \right] \right\}. \quad (32)$$

Likewise, we have

$$I_3(u, v_0, \sigma) \approx -\frac{\sqrt{4 - \sigma^2}}{2uv_0\sigma} \left[(\alpha - \beta) \left(1 + \frac{\sigma^2}{8}\right) + s(\alpha^2 - \beta^2) \left(1 - \frac{\sigma^2}{8}\right) - \frac{\sigma^2(1 + \Sigma^2)}{8} \times \left(\frac{\alpha}{1 - \alpha\Sigma} - \frac{\beta}{1 - \beta\Sigma} \right) \right]. \quad (33)$$

Relations (32) and (33) yield

$$F_2(u, v_0, \sigma) \approx 1 + \frac{2u^2}{\pi v_0} I_3(u, v_0, \sigma) + \text{Im} \left[I_2 \left(A_1, \frac{u + v_0}{\sqrt{2u}}, + \frac{v_0\sigma}{\sqrt{2u}}, - \frac{\pi}{4} \right) \right] + \text{Re} \left[I_2 \left(A_2, \frac{u + v_0}{\sqrt{2u}}, + \frac{v_0\sigma}{\sqrt{2u}}, - \frac{\pi}{4} \right) \right] + \text{Im} \left[I_2 \left(A_3, \frac{u - v_0}{\sqrt{2u}}, - \frac{v_0\sigma}{\sqrt{2u}}, + \frac{\pi}{4} \right) \right] + \text{Re} \left[I_2 \left(A_4, \frac{u - v_0}{\sqrt{2u}}, - \frac{v_0\sigma}{\sqrt{2u}}, + \frac{\pi}{4} \right) \right]. \quad (34)$$

This relation becomes quite accurate for the combination of a short wavelength and large σ , the case for which quadrature is most difficult. Therefore quadrature and use of the above result constitute complementary techniques.

B. Large Defining Aperture

The same can be said about the result derived in this subsection that has been said about relation (34). The same function I_3 is useful, as is a functional $I_1[B(x), a]$, which is defined in Eq. (A7) below and for which an explicit expression valid for large $|a|$ is given in relation (A13). We can reexpress the results of

relation (20) in the abbreviated form for the integrand in Eq. (23) as follows:

$$\frac{\{(1 - x^2)[(2 + \sigma x)^2 - \sigma^2]\}^{1/2} 2v_0^2 I(u, v_0, 1 + \sigma x)}{1 + \sigma x} \approx \frac{\{(1 - x^2)[(2 + \sigma x)^2 - \sigma^2]\}^{1/2}}{1 + \sigma x} + \frac{2v_0\{(1 - x^2)[(2 + \sigma x)^2 - \sigma^2]\}^{1/2}}{\pi[u^2 - v_0^2(1 + \sigma x)^2]} + \sqrt{1 - x^2} B(x) \cos(2v_0 + 2v_0\sigma x), \quad (35)$$

where we have

$$B(x) = \frac{[(2 + \sigma x)^2 - \sigma^2]^{1/2}}{\pi[v_0^2(1 + \sigma x)^2 - u^2](1 + \sigma x)}. \quad (36)$$

Therefore we have

$$F_1(u, v_0, \sigma) \approx 1 + \frac{2v_0}{\pi} I_3(u, v_0, \sigma) + \cos(2v_0) \text{Re}[I_1(B, 2v_0\sigma)] - \sin(2v_0) \text{Im}[I_1(B, 2v_0\sigma)]. \quad (37)$$

C. Pinhole Defining Aperture

To find F_1 in the case of a pinhole defining aperture we can use either the $u \rightarrow 0$ limit of Subsection 4.B if $v_0(1 - \sigma)$ is sufficiently large or else the suggested numerical quadrature, based on applying relations (21) and (22) to Eq. (23).

5. Demonstration Applications

One example of the diffraction effects of nonlimiting apertures is taken from a set of geometries considered by Boivin.⁸ The merits of using toothed apertures (instead of circular apertures) to reduce diffraction effects were demonstrated there and subsequently modeled in Ref. 9. Here we consider seven of the geometries with circular apertures, which are specified in Table 1. Resultant values of F_2 are shown (for $\lambda = 0.58 \mu\text{m}$) in Table 2. The results in the second column of Table 2 were found from the standard approximation⁴:

$$F_2 \approx 1 + \frac{2}{\pi v_0}. \quad (38)$$

Table 1. Parameters That Specify the Geometries Used in Ref. 8

Geometry	Values of These Parameters				
	r_s (mm)	d_s (mm)	R (mm)	r_d (mm)	d_d (mm)
1	0.5	500	3.5	1.25	500
2	0.5	500	7.5	1.25	500
3	0.5	850	3.5	1.60	400
4	0.5	500	3.5	1.25	850
5	0.5	650	3.5	1.25	700
6	0.5	800	3.5	1.25	550
7	0.5	950	3.5	1.25	400

Table 2. F_2 for Geometries Indicated in Table 1 at $\lambda = 0.58 \mu\text{m}^a$

Geometry	Values of F_2 Found for These Cases				
	Relation (38)	Fresnel			Detailed Calculation
		Point Source	Extended Source	Relation (34)	
1	1.00672	1.00692	1.00683	1.00683	1.00682
2	1.00313	1.00316	1.00309	1.00309	1.00310
3	1.00420	1.00465	1.00464	1.00461	1.00464
4	1.0114	1.0116	1.0110	1.0109	1.0110
5	1.0094	1.0098	1.0095	1.0095	1.0095
6	1.0074	1.0078	1.0077	1.0076	1.0077
7	1.0054	1.0057	1.0057	1.0058	1.0057

^aFound by use of relation (38), for a point source and for the extended source in the Fresnel approximation, by use of relation (34), and as found in more-detailed numerical approximations.

The third and fourth columns show F_2 obtained in the Fresnel approximation, given a point source and the actual extended source, respectively. However, results for the point source omit the effects of the last two terms in the expression $v < u$ in Eq. (8). These effects are large and rapidly oscillatory with λ , so they are largely self-canceling for broadband sources as well as for extended sources. The fifth column shows results found from relation (34). Finally, the last column shows results obtained in two separate calculations that went beyond the Fresnel approximation, one of which is found in Ref. 9. The efficacy of relation (34) is clearly demonstrated.

Likewise, recent measurements at the National Institute of Standards and Technology of black-body radiance have involved a fully illuminated, ≈ 4.516 -mm-radius defining aperture separated by 1116.56 mm from a 14.97-mm-radius defining aperture placed in front of a radiometer. To reduce stray light and for other reasons, two 15.835-mm-radius baffles were placed 494.84 mm (Baffle A) and 960.98 mm (Baffle B) from the 4.516-mm-radius aperture. Some diffraction loss was incurred because of this aperture, but we now consider only the diffraction effects of the baffles. To do this, we treat the 4.516-mm-radius aperture as an extended Lambertian source. For Baffle A, relation (34) has proved useful for all $\lambda \leq 50 \mu\text{m}$, the longest wavelength that we consider. For Baffle B, relation (34) breaks down for larger λ in this range, but 12-point quadrature has proved adequate at such wavelengths. In Fig. 2 the results for the relative excess flux divided by the wavelength are shown for both baffles. Results found from calculations done in the Fresnel approximation are shown by solid curves. This approximation works adequately for the geometries considered. Results from relation (34) (long-dashed curve) and from the quadrature (short-dashed curve) are also shown.

Steel *et al.*³ analyzed F_1 for situations involving extended sources that modeled diffraction in geometries used by Blevin and Brown.¹⁰ Nine combinations of parameters u , v_0 , and $v_0\sigma$ (to use the present

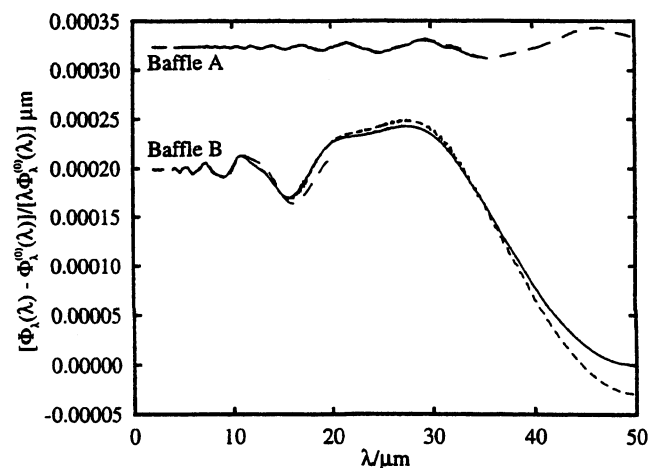


Fig. 2. $[\Phi_\lambda(\lambda) - \Phi_\lambda^{(0)}(\lambda)]/[(\lambda/\mu\text{m})\Phi_\lambda^{(0)}(\lambda)]$, or the relative excess spectral flux divided by wavelength that results from two baffles discussed in the text, as found by numerical calculations carried out in the Fresnel approximation (solid curves), from relation (34) (long-dashed curve), and from 12-point quadrature (short-dashed curve).

notation) were considered. These are listed in Table 3. Also shown are values for F_1 obtained from relation (20) and the formula of Steel *et al.*:

$$F_1(u, v_0, v_0\sigma) \approx 1 - \frac{1}{2\pi v_0\sigma} \ln \left[\frac{v_0^2(1 + \sigma)^2 - u^2}{v_0^2(1 - \sigma)^2 - u^2} \right] \quad (39)$$

and as found from relation (37) and numerical calculations done in the Fresnel approximation. The last three results agree adequately, with relation (37) adhering most closely to the detailed numerical results.

Diffraction in one recently encountered setup involving pinholes¹¹ could be modeled by consideration of a 1.778-mm-radius extended source separated by 150 mm from a 6.35-mm-radius detector. Pinhole defining apertures with radii of 52 and 101 μm were placed 23 mm from the source, and diffraction losses because of these apertures were of concern. The resultant values of F_1 found by 12-point Gauss-Chebyshev quadrature and relations (21) and (22) are

Table 3. Geometry Parameters and F_1 for Nine Examples of Limiting Apertures^a

Geometry Parameter			Values of F_1 for These Cases			
u	v_0	$v_0\sigma$	Relation (20)	Relation (39)	Relation (37)	Numerical
65	153	35	0.99492	0.99470	0.99470	0.99470
84	299	35	0.99769	0.99767	0.99767	0.99766
216	940	35	0.99929	0.99928	0.99928	0.99929
113	202	46	0.99541	0.99502	0.99506	0.99506
146	394	46	0.99812	0.99811	0.99811	0.99811
376	1240	46	0.99943	0.99943	0.99943	0.99944
175	252	57	0.99512	0.99394	0.99417	0.99422
225	490	57	0.99836	0.99833	0.99833	0.99834
582	1540	57	0.99952	0.99952	0.99952	0.99952

^aResults are presented as found for relations (20), (39), (37), and numerical results carried out in the Fresnel approximation.

Table 4. F_1 for Two Apertures Found from Simplified Formulas Discussed in the Text and Detailed Calculations Using the Fresnel Approximation for Two Different Pinhole Apertures R

λ (μm)	$F_1, R = 52 \mu\text{m}$		$F_1, R = 101 \mu\text{m}$	
	This Study	Fresnel	This Study	Fresnel
5	0.848	0.847	0.921	0.920
10	0.650	0.648	0.843	0.841
15	0.449	0.449	0.742	0.740
20	0.305	0.305	0.637	0.635
25	0.214	0.214	0.530	0.529
30	0.157	0.157	0.433	0.433

listed in Table 4, as are values of F_1 found by detailed calculations with the Fresnel approximation. In particular, the ratio of the spectral flux incident upon the detector for the two pinholes was of interest, and

Appendix A

Equation (12) can be derived as follows: We simplify Eq. (10) by considering the fraction of a circle with radius τ that is inside a unit circle when the distance between the circles' centers is σ' . Define this fraction as $C(\sigma', \tau)$, which is normalized so that $C(\sigma', \tau)$ is 1 when the radius τ circle is wholly contained within the unit circle. Equation (10) can be rewritten as

$$\Phi_\lambda(\lambda) = \frac{2\pi r_s^2 r_d^2}{(d_s + d_d)^2} \frac{L_\lambda(\lambda)}{\sigma^2} \int_0^\infty d\tau \tau |\alpha(u, v_0 \tau)|^2 E(\sigma, \tau), \quad (\text{A1})$$

where $E(\sigma, \tau)$ is defined by

$$E(\sigma, \tau) = 2\pi \int_0^\sigma d\sigma' \sigma' C(\sigma', \tau), \quad (\text{A2})$$

and it is given by

$$E(\sigma, \tau) = \begin{cases} \pi\sigma^2 & \tau < 1 - \sigma \\ \pi(1 - \tau)^2 + 2 \int_{1-\tau}^\sigma d\sigma' \sigma' \arccos\left(\frac{\sigma'^2 + \tau^2 - 1}{2\tau\sigma'}\right) & 1 - \sigma \leq \tau < 1 \\ 2 \int_{\tau-1}^\sigma d\sigma' \sigma' \arccos\left(\frac{\sigma'^2 + \tau^2 - 1}{2\tau\sigma'}\right) & 1 \leq \tau < 1 + \sigma \\ 0 & \tau \geq 1 + \sigma \end{cases} \quad (\text{A3})$$

calculated results shown here mimicked experiment in a reasonable fashion¹¹ but with deviations toward longer wavelength, perhaps because of finite aperture thickness.

6. Conclusions

Diffraction by a circular aperture has been analyzed, with the explicit goal of determining diffraction effects on the spectral flux incident upon a detector for the cases of point and extended sources. Toward this goal, approximate expressions describing the effects of diffraction have been obtained. In many instances these expressions have been given in closed form, whereas in other circumstances they have been given in a form whose evaluation requires a practitioner to perform one simple integration as indicated. Whereas the resultant formulas are slightly more complicated than those presented in the past, they should also have a considerably wider applicability. The results should be most helpful in cases of short wavelength and are therefore complementary to the results obtained in the long-wavelength case within the geometrical theory of diffraction.¹² Formulas have been presented for cases of large nonlimiting apertures, large limiting apertures, and pinhole limiting apertures, and their efficacy was assessed for several geometries of interest in practical radiometry.

It is convenient to write

$$\begin{aligned} & \int_0^\infty d\tau \tau |\alpha(u, v_0 \tau)|^2 E(\sigma, \tau) \\ &= \int_0^\infty d\tau \left[\frac{dI(u, v_0, \tau)}{d\tau} \right] E(\sigma, \tau) \\ &= \int_0^\infty d\tau I(u, v_0, \tau) \left[- \frac{dE(\sigma, \tau)}{d\tau} \right], \end{aligned} \quad (\text{A4})$$

where Eq. (11) and integration by parts have been used, exploiting the fact that

$$I(u, v_0, 0) = E(\sigma, \infty) = 0. \quad (\text{A5})$$

Differentiating Eq. (A3) with respect to τ and evaluating necessary integrals, we find that

$$\begin{aligned} & - \frac{dE(\sigma, \tau)}{d\tau} \\ &= \begin{cases} \frac{\sigma}{\tau} \{(1 - x^2)[(2 + \sigma x)^2 - \sigma^2]\}^{1/2} & 1 - \sigma < \tau < 1 + \sigma \\ 0 & \text{otherwise} \end{cases} \end{aligned} \quad (\text{A6})$$

where x is implicitly defined by $\tau = 1 + \sigma x$. This result can also be deduced from geometrical considerations.

Approximate expressions for integrals in relations (34) and (37) can be arrived at as follows: To obtain I_1 for any smooth $f(x)$ and $|a| \gg 1$, rewrite it as

$$\begin{aligned} I_1[f, a] &= \frac{1}{\pi} \int_{-1}^1 dx \sqrt{1-x^2} \exp(iax) f(x) \\ &= \frac{1}{\pi} \sum_{m=0}^{\infty} (2 - \delta_{m0}) \int_0^{\pi} d\theta \cos(m\theta) i^m J_m(a) \\ &\quad \times (1 - x^2) f(x), \end{aligned} \quad (\text{A7})$$

with θ and x related by $x = \cos \theta$. With the expansion

$$D(\theta) = (1 - x^2) f(x) = f(x) \sin^2 \theta = \sum_{m=0}^{\infty} \cos(m\theta) D_m, \quad (\text{A8})$$

Eq. (A7) is rewritten as

$$I_1[f, a] = \sum_{m=0}^{\infty} i^m J_m(a) D_m. \quad (\text{A9})$$

Analysis of $D(\theta)$ yields

$$\begin{aligned} \sum_{m=0}^{\infty} (\pm 1)^m D_m &= [(1 - x^2) f(x)]_{x=\pm 1} = 0, \quad (\text{A10}) \\ \sum_{m=0}^{\infty} (\pm 1)^m m^2 D_m &= \left\{ -\frac{\partial^2}{\partial \theta^2} [f(x) \sin^2 \theta] \right\}_{x=\pm 1} \\ &= -2f(\pm 1). \end{aligned} \quad (\text{A11})$$

Likewise, expansion of $J_m(|a|)$ for large $|a|$ yields

$$\begin{aligned} J_m(|a|) &\approx \sqrt{\frac{2}{\pi|a|}} \left[\cos\left(|a| - \frac{m\pi}{2} - \frac{\pi}{4}\right) \right. \\ &\quad \left. - \frac{(m^2 - 1/4)}{2|a|} \sin\left(|a| - \frac{m\pi}{2} - \frac{\pi}{4}\right) \right]; \end{aligned} \quad (\text{A12})$$

the remaining terms can be neglected for sufficiently large $|a|$. On substitution of relation (A12) into Eq. (A9), the sum rule [Eq. (A10)] causes contributions with prefactors of order m^0 to cancel in I_1 , whereas Eq. (A11) renders a simple expression for contributions with prefactors of order m^2 . Using $I_1[f, a] = I_1^*[f, -a]$, we obtain, for all $|a| \gg 1$,

$$\begin{aligned} I_1[f, a] &\approx \frac{1}{2|a|} \sqrt{\frac{2}{\pi|a|}} \left\{ \sin\left(|a| - \frac{\pi}{4}\right) [f(1) + f(-1)] \right. \\ &\quad \left. - i \frac{a}{|a|} \cos\left(|a| - \frac{\pi}{4}\right) [f(1) - f(-1)] \right\}. \end{aligned} \quad (\text{A13})$$

This result is helpful when one is evaluating (for smooth g)

$$\begin{aligned} I_2[g, a, b, c] &= \frac{1}{\pi} \int_{-1}^1 dx \sqrt{1-x^2} \\ &\quad \times \exp\{i[(a+bx)^2 + c]\} g(x). \end{aligned} \quad (\text{A14})$$

To that end, change the variable of integration to

$$\alpha = (a + bx)^2, \quad (\text{A15})$$

so α and x are further related by

$$x = \frac{\alpha^{1/2} - a}{b}, \quad dx = \frac{d\alpha}{2b\alpha^{1/2}}, \quad (\text{A16})$$

which yields

$$I_2[g, a, b, c] = \frac{e^{ic}}{2\pi b} \int_{(a-b)^2}^{(a+b)^2} \frac{d\alpha}{\alpha^{1/2}} \sqrt{1-x^2} e^{ia} g(x). \quad (\text{A17})$$

Then, define x' so that α and x' are related by

$$\alpha = a^2 + b^2 + 2abx', \quad x' = \frac{\alpha - a^2 - b^2}{2ab}. \quad (\text{A18})$$

I_2 simplifies to

$$\begin{aligned} I_2[g, a, b, c] &= \exp[i(a^2 + b^2 + c)] \frac{1}{\pi} \int_{-1}^1 dx' \sqrt{1-x'^2} \\ &\quad \times \exp(2iabx') f(x') \\ &= \exp[i(a^2 + b^2 + c)] I_1[f, 2ab], \end{aligned} \quad (\text{A19})$$

where $f(x')$ is given by

$$f(x') = \frac{ag(x)}{\alpha^{1/2}} \left(\frac{1-x^2}{1-x'^2} \right)^{1/2}. \quad (\text{A20})$$

In particular, we have

$$f(\pm 1) = \left(\frac{a}{a \pm b} \right)^{3/2} g(\pm 1). \quad (\text{A21})$$

The resulting I_2 is valid for all $|2ab| \gg 1$ that satisfy $a > |b|$.

Using the values of a' and b' specified by Eqs. (29) permits the integral in Eq. (31) to be rewritten as follows: By the relations

$$\begin{aligned} \frac{1}{u^2 - v_0^2(1 + \sigma x)^2} &= \frac{1}{(u^2 - v_0^2)(a' - b')} \\ &\quad \times \left(\frac{a'}{1 - a'x} - \frac{b'}{1 - b'x} \right) \\ &= -\frac{1}{2uv_0\sigma} \left(\frac{a'}{1 - a'x} - \frac{b'}{1 - b'x} \right), \end{aligned} \quad (\text{A22})$$

we can rewrite

$$I_3[u, v_0, \sigma] = \frac{1}{\pi} \int_{-1}^1 dx \frac{\{(1-x^2)[(2+\sigma x)^2 - \sigma^2]\}^{1/2}}{u^2 - v_0^2(1+\sigma x)^2} \\ = -\frac{K(a') - K(b')}{2uv_0\sigma}, \quad (\text{A23})$$

where K is defined by

$$K(a') = \frac{1}{\pi} \int_{-1}^1 dx \frac{a'\{(1-x^2)[(2+\sigma x)^2 - \sigma^2]\}^{1/2}}{1-a'x} \\ = \frac{2\alpha}{\pi} \int_{-1}^1 dx \{(1-x^2)[(2+\sigma x)^2 - \sigma^2]\}^{1/2} \\ \times \sum_{n=0}^{\infty} \alpha^n \Xi_n(x). \quad (\text{A24})$$

Here Ξ_n denotes a Chebyshev polynomial of the second kind, and α is given by Eqs. (29).

Using the shorthand

$$y = sx = \frac{\sigma x}{4 - \sigma^2}, \quad (\text{A25})$$

where s is defined in Eqs. (29), we can rewrite

$$[(2+\sigma x)^2 - \sigma^2]^{1/2} = \{(4-\sigma^2)(1+2y)^2 \\ - \sigma^2 y^2\}^{1/2} = \sqrt{4-\sigma^2} \\ \times \left(1 + 2y - \frac{\sigma^2 y^2/2}{1+2y} + \dots\right). \quad (\text{A26})$$

Additional terms in the rightmost expression are weighted by ascending powers of σ^4 . Therefore retaining only the terms shown is usually an acceptable approximation.

Rearrangement of terms in the right-most expression gives

$$[(2+\sigma x)^2 - \sigma^2]^{1/2} \approx \sqrt{4-\sigma^2} \left[\left(1 + \frac{\sigma^2}{8}\right) \Xi_0(x) \right. \\ \left. + s \left(1 - \frac{\sigma^2}{8}\right) \Xi_1(x) - \frac{\sigma^2}{8} \right. \\ \left. \times (1 + \Sigma^2) \sum_{n=0}^{\infty} \Sigma^n \Xi_n(x) \right], \quad (\text{A27})$$

where Σ is given by Eqs. (29). Substitution of relation (A27) into Eq. (A24) yields

$$K(a') \approx \alpha \sqrt{4-\sigma^2} \left[(1 + \sigma^2/8) + s\alpha(1 - \sigma^2/8) \right. \\ \left. - \frac{\sigma^2(1 + \Sigma^2)/8}{1 - \alpha\Sigma} \right]. \quad (\text{A28})$$

Using the analogous expression for $K(b')$ completes the derivation of relation (33). The utility of using relation (A27) is aided by the $(1-x^2)^{1/2}$ weight in Eq. (31), because the approximation made is the poorest for low x . This approximation becomes exact for $\sigma \rightarrow 0$ and degrades with increasing σ . One may have σ as large as 1 for a nonlimiting aperture, whereas $\sigma \rightarrow 1$ in the case of a limiting aperture can occur only in the case of a pinhole aperture. In that case relation (33) may be unsuitable anyhow. For a nonlimiting aperture, $\sigma = 1$ implies $u \geq 2v_0$. In that case, when relation (33) is used the relative error in I_3 is at most ~ 0.01 .

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