Characteristic Impedance, Power, and Causality

Dylan F. Williams, *Senior Member, IEEE*, and Bradley K. Alpert, *Member, IEEE*

***Abstract*—A new causal power-normalized waveguide equivalent-circuit theory determines uniquely both the magnitude and phase of the characteristic impedance of a waveguide.**

***Index Terms*—Causality, characteristic impedance, microwave circuit theory.**

I. Introduction

We SUMMARIZE the causal waveguide circuit theory of [1][1] and show that it determines uniquely the characteristic impedance of a single-mode waveguide. The theory marries the power normalization of [2] with additional constraints that enforce simultaneity of the theory's voltages and currents and the actual fields in the circuit. These additional constraints guarantee that the network parameters of passive devices in this theory are causal. We will show that the new theory requires Z_0 be real and constant in a lossless coaxial waveguide. We will also show that in lossless rectangular waveguide, Z_0 must be proportional to the wave impedance: the choice $|Z_0| = 1$ is not allowed.

II. The Causal Circuit Theory

The equivalent-circuit theory of [1] begins with a waveguide that is uniform in the axial direction and supports only a single mode of propagation at the reference plane where v and i are defined. The voltage v is defined by

$$\boldsymbol{E}_{\rm t}(\boldsymbol{r}, z) = [c_+ e^{-\gamma z} + c_- e^{\gamma z}]\boldsymbol{e}_{\rm t}(\boldsymbol{r}) \equiv \frac{v(z)}{v_0}\boldsymbol{e}_{\rm t}(\boldsymbol{r}) \qquad (1)$$

and the current i by

$$\boldsymbol{H}_{\rm t}(\boldsymbol{r}, z) = [c_+ e^{-\gamma z} - c_- e^{\gamma z}]\boldsymbol{h}_{\rm t}(\boldsymbol{r}) \equiv \frac{i(z)}{i_0}\boldsymbol{h}_{\rm t}(\boldsymbol{r}) \qquad (2)$$

where

$\boldsymbol{r} = (x, y)$	transverse coordinate;
$\boldsymbol{E}_{\rm t}$ and $\boldsymbol{H}_{\rm t}$	total transverse electric and magnetic fields in the guide;
$\boldsymbol{e}_{\rm t}$ and $\boldsymbol{h}_{\rm t}$	transverse modal electric and magnetic fields of the single propagating mode;
γ	modal propagation constant;
c_+ and c_-	forward and reverse amplitudes of the mode.

Manuscript received January 7, 1999.

The authors are with the National Institute of Standards and Technology, Boulder, CO 80303 USA (e-mail: dylan@boulder.nist.gov).

Publisher Item Identifier S 1051-8207(99)04713-3.

[1] Advance copies available at http://www.boulder.nist.gov/div813/dylan/ and http://math.nist.gov/mcsd/Staff/BAlpert/.

The time dependence $e^{j\omega t}$ in (1) and (2) have been suppressed, all of the parameters are functions of ω, and the longitudinal components of the fields are given explicitly in [1]. The two factors v_0 and i_0 define v and i in terms of the fields and can be thought of as voltage and current normalization factors.

The total time-averaged power p in the waveguide is found by integrating the Poynting vector over the guide's cross-section S

$$p \equiv \frac{1}{2}\int_S \boldsymbol{E}_{\rm t} \times \boldsymbol{H}_{\rm t}^* \cdot \boldsymbol{z}\, dS = \frac{1}{2}\frac{vi^*}{v_0 i_0^*}\int_S \boldsymbol{e}_{\rm t} \times \boldsymbol{h}_{\rm t}^* \cdot \boldsymbol{z}\, dS. \qquad (3)$$

The power normalization of [2] is achieved by imposing the constraint

$$v_0 i_0^* = p_0 \equiv \int_S \boldsymbol{e}_{\rm t} \times \boldsymbol{h}_{\rm t}^* \cdot \boldsymbol{z}\, dS \qquad (4)$$

which ensures that the time-averaged power is $p = \frac{1}{2}vi^*$.

III. Characteristic Impedance

The characteristic impedance Z_0 of a waveguide is defined by the ratio of v to i when only the forward mode is present [2]

$$Z_0 \equiv \left.\frac{v}{i}\right|_{c_-=0} = \frac{v_0}{i_0} = \frac{|v_0|^2}{p_0^*} = \frac{p_0}{|i_0|^2}. \qquad (5)$$

Equation (5) shows that the power normalization requires $\arg(Z_0) = \arg(p_0)$, which from (4) is a fixed property of the guide uniquely determined by the modal field solutions $\boldsymbol{e}_{\rm t}$ and $\boldsymbol{h}_{\rm t}$.

The causal circuit theory of [1] requires that both $Z_0(\omega)$ and $Y_0(\omega) \equiv 1/Z_0(\omega)$ be causal. That is, the theory requires that $\hat{Z}_0(t) = \hat{Y}_0(t) = 0$ for $t < 0$, where $\hat{Z}_0(t)$ and $\hat{Y}_0(t)$ are the inverse Fourier transforms of $Z_0(\omega)$ and $Y_0(\omega)$.

These conditions ensure that the waveguide responds to input signals after, not before, their onset. They also imply that $Z_0(\omega)$ is minimum phase [3].

The minimum phase constraint is a strong one. It implies that the real and imaginary parts of the complex logarithm of a minimum phase function are a Hilbert transform pair: that is, $\arg(Z_0)$ must be equal to the Hilbert transform of $\ln|Z_0|$ [3]. As a result, we can determine $\ln|\lambda Z_0|$, where λ is a constant, from $\arg(Z_0)$ [1].

IV. Lossless Coaxial Transmission Line

The power flow p_0 is real in a lossless coaxial transmission line, so the phase of Z_0 is zero. The set of constant functions form the null space of the Hilbert transform, so in the causal circuit theory of [1] Z_0 must be real and constant.

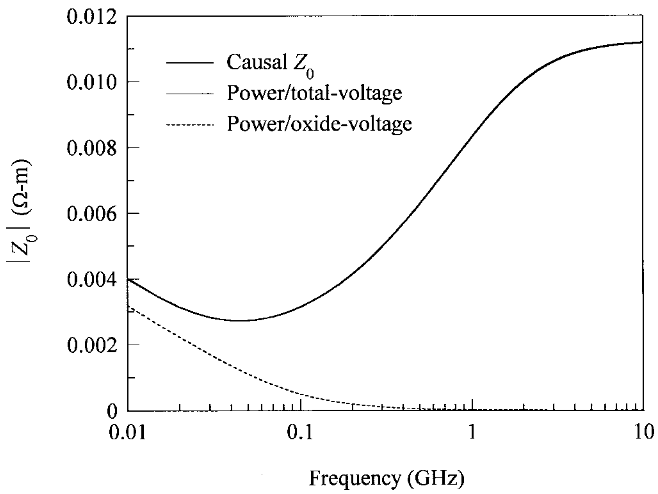

Fig. 1. $|Z_0|$ for the MIS transmission line of [5]. The two solid curves are so close as to be indistinguishable.

V. Dominant TE_{10} Mode of Lossless Rectangular Waveguide

The power flow p_0 and therefore Z_0 of the TE_{10} mode of a lossless rectangular waveguide are real above its cutoff frequency ω_c and imaginary below ω_c. So $\arg(Z_0)$ is equal to $\pm\pi/2$ below ω_c, and 0 above. The Hilbert transform of

$$\frac{1}{2}\ln\left|\frac{\omega^2}{\omega^2-\omega_c^2}\right| \tag{6}$$

is equal to $-\pi/2$ for $-\omega_c < \omega < 0$, $\pi/2$ for $0 < \omega < \omega_c$, and 0 elsewhere [4]. So the causality constraints of [1] require that

$$|Z_0| \propto \sqrt{\left|\frac{\omega^2}{\omega^2-\omega_c^2}\right|} \tag{7}$$

where $\propto$ indicates proportionality. That is, Z_0 must be proportional to the wave impedance of the guide: the choice $|Z_0| = 1$ is not admissible in the causal theory.

VI. MIS Transmission Line

Different choices of voltage and current paths in conventional waveguide circuit theories result in different characteristic impedances in metal–insulator–semiconductor (MIS) transmission lines. Not all of these choices are consistent with causality.

Fig. 1 compares three characteristic impedances for the TM_{01} mode of the infinitely wide MIS line investigated in [5]. This MIS line consists of a 1.0-μm-thick metal signal plane with a conductivity of 3×10^7 S/m separated from the 100-μm-thick 100-Ω-cm silicon supporting substrate by a 1.0-μm-thick oxide with conductivity of 10^{-3} S/m. The ground conductor on the back of the silicon substrate is infinitely thin and perfectly conducting.

The two solid curves in Fig. 1, which are labeled "Causal Z_0" and "Power/total-voltage," agree so closely as to be indistinguishable on the graph. The curve "Causal Z_0" is the magnitude of the characteristic impedance determined from the phase of p_0 and the minimum phase properties of Z_0, in accordance with the causal theory of [1]. The curve "Power/total-voltage" is the magnitude of the characteristic impedance defined with a power-voltage definition. Here the power normalization is based on (4) (the integral of the Poynting vector over the guide cross section) and the voltage normalization on

$$v_0 = -\int_{\text{path}} \boldsymbol{e}_t \cdot dl \tag{8}$$

where the path begins at the ground on the back of the silicon substrate and terminates on the conductor metal on top of the oxide. This normalization implies that

$$v = -\int_{\text{path}} \boldsymbol{E}_t \cdot dl. \tag{9}$$

Conventional circuit theories do not specify voltage path uniquely, and a voltage path in the MIS line from the silicon surface through the oxide to the signal line is equally consistent with those theories. However, Fig. 1 shows that the characteristic impedance defined from the power constraint of (4) and the voltage across the oxide, which is labeled "Power/oxide-voltage," differs significantly from the characteristic impedance required by the causal theory of [1]. Further investigation [1] shows that $\hat{Z}_0(t)$ defined from the oxide voltage is indeed nonzero for $t < 0$.

VII. Conclusion

We have studied some of the implications of the causal power-normalized waveguide circuit theory of [1]. The examples illustrate an important contribution of the causal theory: it replaces the subjective and sometimes misleading "common-sense" criteria for defining Z_0 with a clear and unambiguous procedure that guarantees causal responses. This new approach should be especially useful in complex transmission structures where the choice of voltage and current paths are not intuitively obvious.

References

[1] D. F. Williams and B. K. Alpert, "Causality and waveguide circuit theory," *IEEE Trans. Microwave Theory Tech.*, submitted for publication.
[2] R. B. Marks and D. F. Williams, "A general waveguide circuit theory," *J. Res. Natl. Inst. Stand. Technol.*, vol. 97, no. 5, pp. 533–562, Sept./Oct. 1992.
[3] E. A. Guillemin, *Synthesis of Passive Networks.* New York: Wiley, 1957.
[4] Bateman Manuscript Project, *Tables of Integral Transforms.* New York: McGraw-Hill, 1954, Table 15.2 (42).
[5] D. F. Williams, "Metal–insulator–silicon transmission line model," in *51st ARFTG Conf. Dig.*, June 12, 1998, pp. 65–71.