Calibration of ac susceptometer for cylindrical specimens

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The absolute magnetic susceptibility of cylindrical specimens is obtained with an ac susceptometer whose calibration is based on a calculation of mutual inductance. An axially magnetized cylinder is modeled as a solenoid of the same size. The mutual inductance between such a solenoid and a pickup coil of arbitrary dimensions is computed. The susceptibility is then a function of the mutual inductance, the cylinder length, the magnitude and frequency of the ac magnetizing field, and the voltage induced on the pickup coil. Demagnetization factor and eddy-current effects are considered, an example is given, and pickup coil compensation is discussed. Other calibration methods are also presented.

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INTRODUCTION

A useful method of measuring the magnetic susceptibility of nonferromagnetic materials employs an ac field source and a pickup coil. A typical circuit is shown in Fig. 1. Such pickup coils usually require calibration with standards of similar size to that of the specimens under study, except for the case of thin pickup coils with ellipsoidal samples, whose dipolar fields allow a closed-form mathematical solution.[1]

A method of calibration of an ac susceptometer for relatively long cylindrical specimens of arbitrary dimensions, coaxial with a pickup coil, is based on a calculation of mutual inductance. The mutual inductance between two coaxial coils (inner and outer) may be numerically computed. Since the flux density distribution of a uniformly, longitudinally magnetized cylinder of uniform composition is the same as that of a solenoid of the same size, a cylindrical specimen may be modeled as an inner coil. The calculation of mutual inductance leads to the absolute magnetic volume susceptibility of the cylinder, when an ac field is applied, in terms of the emf induced on the outer pickup coil.

I. CALIBRATION BY CALCULATION OF MUTUAL INDUCTANCE

A. Equivalent solenoid

In general, the flux density **B** may be thought of as due to Amperian currents of volume density $\nabla \times \mathbf{M}$ and surface density $\mathbf{M} \times \hat{n}$, where **M** is the magnetization and $\hat{n}$ is the unit vector normal to the surface. The flux density **B** external to a cylinder of uniform axial magnetization ($\nabla \times \mathbf{M} = 0$) is equivalent to that of a model solenoid of the same size with ampere-turn density NI/l, where M has units A/m, N is the number of turns, I is the current, and l is the length:

$$M = NI/l. \tag{1}$$

The near field cannot be expressed in closed form, but may be computed by superposition of **B** due to elemental uniformly magnetized disks or elemental circular current loops for the cylinder or equivalent solenoid, respectively. The near field of a current loop is well known.[2,3]

The total magnetic flux (flux linkage) Φ sensed by the pickup coil may be calculated from either **B** or the vector potential **A**:

$$\Phi = \iint \mathbf{B} \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{s}, \tag{2}$$

where $d\mathbf{a}$ is the incremental area, $d\mathbf{s}$ is the incremental contour, and the integrations are over the entire pickup coil. The mutual inductance per solenoid turn is

$$L^* = L_{12}/N = \Phi/NI = \frac{1}{NI} \oint \mathbf{A} \cdot d\mathbf{s}, \tag{3}$$

where L_{12} is the mutual inductance between the solenoid and the pickup coil. L_{12} is proportional to N; L^* is independent of N but depends on the geometries of the solenoid (i.e., specimen) and the pickup coil, and the number of turns on the pickup coil. A general mutual inductance calculation is detailed in Sec. I B. The emf induced in the pickup coil due to the solenoid is

$$v = -d\Phi/dt = -L_{12}\, dI/dt. \tag{4}$$

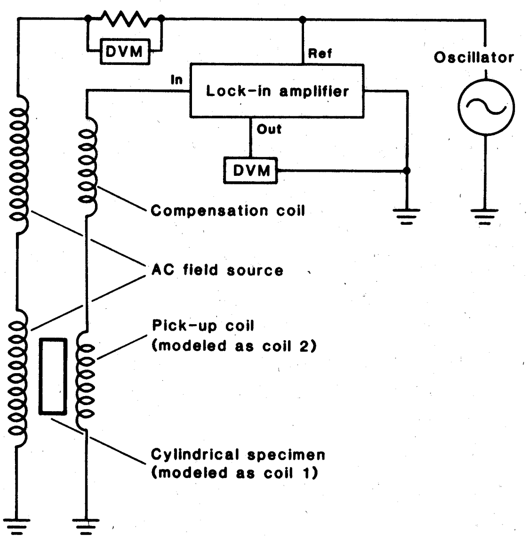

FIG. 1. One method [see number (6) in Table II] of measuring ac susceptibility.

Converting current to magnetization using Eq. 1,

$$v = -L^{*} l\, dM/dt. \tag{5}$$

To this point, the ac field source has not been part of the derivation. In fact, *it has been and will continue to be assumed that the emf induced in the pickup coil by the field source is nulled* (see Sec. II B). The ac field is assumed to be uniform. Substituting $M = \chi H$, where χ is the volume susceptibility (dimensionless), $H = H_0 \cos 2\pi f t$ is the applied ac magnetic field, and rearranging Eq. (5), the final result is obtained:

$$\chi = \frac{v_{\rm rms}}{L^{*} l\, 2\pi f H_{\rm rms}}, \tag{6}$$

where f is in hertz and χ is in SI units. To convert to cgs units, the value of χ is divided by 4π. The assumption of uniform M and H is a good one for relatively long cylinders characterized by small demagnetiztion factors (see Sec. I C).

B. Mutual inductance of coaxial solenoids

The mutual inductance [L_{12} in Eq. (3)] may be calculated using computer techniques for the general case of coaxial, but not necessarily concentric, thick solenoids. Figure 2 shows the geometry and defines some of the variables. The mutual inductance is calculated by integrating **A**, due to a current density J in coil 1, around the turns of coil 2. The vector potential **A** has only a ϕ component. At the point (r,ϕ,z) in cylindrical coordinates,[4]

$$\begin{aligned} A_\phi(r,z) &= \frac{\mu_0 J}{4\pi}\int_{a_{11}}^{a_{12}} da\, a \int_0^{2\pi} d\theta \cos\theta \ln[\xi + (\xi^2 + r^2 + a^2 - 2ar\cos\theta)^{1/2}]\Big|_{\xi_1}^{\xi_2} \\ &= \frac{\mu_0 J}{4\pi}\int_{a_{11}}^{a_{12}} da\, a^2 \int_0^{2\pi} d\theta \left(\frac{\xi r \sin^2\theta}{(a^2 + r^2 - 2ar\cos\theta)(\xi^2 + r^2 + a^2 - 2ar\cos\theta)^{1/2}}\right)\Big|_{\xi_1}^{\xi_2} \equiv \frac{\mu_0 J}{4\pi}\Lambda(r,z), \end{aligned} \tag{7}$$

where $\xi = z - \lambda$, $\xi_1 = z - l/2$, $\xi_2 = z + l/2$, and the cylindrical coordinates of each current element are a, θ, λ. J is assumed constant and is simply

$$J = \frac{NI}{l(a_{12} - a_{11})}. \tag{8}$$

From the expression for L_{12} in Eq. (3), and using Eqs. (7) and (8)

$$L_{12} = \frac{\mu_0 N}{4\pi l(a_{12} - a_{11})}\oint \Lambda(r,z)\cdot ds, \tag{9}$$

where I is canceled. For the purpose at hand, $a_{12} \gtrsim a_{11} = d/2$, where d is the specimen diameter. All that remains is to compute the contour integral of Λ.

For purposes of computation, coil 2 is divided into $n \times m$ circular rings, usually not coincident with the actual windings (see Fig. 2). The radius of each ring is

$$r_i = a_{21} + (i - \tfrac{1}{2})(a_{22} - a_{21})/n, \quad i = 1,2,\ldots,n, \tag{10}$$

and its position is

$$z_j = z_d - \tfrac{1}{2}l_2 + (j - \tfrac{1}{2})l_2/m, \quad j = 1,2,\ldots,m. \tag{11}$$

The double integral Λ [given in Eq. (7)] is evaluated for each ring located at (r_i, z_j) using a standard library routine. Then the contour integral of Λ in Eq. (9) is simply

$$\oint \Lambda(r,z)\cdot ds = \frac{N_2}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m} 2\pi r_i\, \Lambda(r_i, z_j), \tag{12}$$

where ds is $r\, d\phi$. The factor N_2/mn is introduced to scale L_{12} to reflect the actual number of turns in coil 2. If the coil displacement z_d is zero (the usual case when the specimen cylinder is centered in the pickup coil), the index j can run from 1 to $m/2$ and the result for L_{12} doubled, thereby taking advantage of the axial symmetry. The number of $n \times m$ discrete segments should be chosen to optimize the tradeoff between accuracy and computation time.

Other procedures may be used to calculate L_{12}. Thin-coil approximations are clearly suitable for the model solenoid and are often adequate for the pickup coil.[5]

C. Demagnetization factor considerations

When susceptibilities are large or when cylinders are not too long, the measured χ should be corrected for demagnetization fields

$$\chi_{\rm int} = \chi/(1 - D\chi), \tag{13}$$

where $\chi_{\rm int}$ is the "internal" susceptibility, corrected for demagnetization, and D is the demagnetization factor ($0 < D < 1$ in SI). Exact values of D are obtainable only for ellipsoids. Average values for uniformly magnetized cylinders have been computed by Brown[6,7] and Crabtree[8] and are given in Table I. These "magnetometric" values are greater than those "ballistic" values reported by Bozorth and Chapin[9] for highly permeable rods. The latter were experimentally determined and based on the magnetization at the middle of a rod rather than an average over the entire volume. Note that the approximation of uniform longitudinal mag-

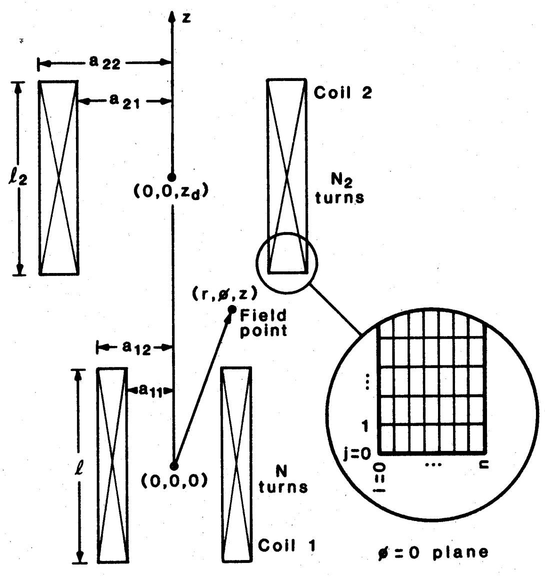

FIG. 2. Geometry for mutual inductance calculations.

netization is realistic only in the cases of materials with small susceptibilities, or relatively long cylinders characterized by small demagnetization factors. Therefore, calculations with cylinders are always approximate to some degree.

D. Eddy current considerations

The eddy currents induced in a conducting material exposed to an alternating magnetic field are a function of the material's susceptibility, resistivity, and geometry. These currents give rise to an internal field which differs from the applied field in magnitude and phase. For accurate susceptibility measurements, these deviations should be minimized.

Zijlstra[10] examines the case of an infinitely long cylinder subject to an ac axial field. The deviations in field magnitude and phase are a function of the reduced radius

$$a_0 = d/\sqrt{2}\delta, \tag{14}$$

where d is the cylinder diameter and δ is the skin depth,

$$\delta = (\rho/\pi f\mu)^{1/2}, \tag{15}$$

where ρ is the material resistivity, f is the frequency, and μ is the permeability equal to $\mu_0(1+\chi)$, where μ_0 is the permeability of free space.

The analysis shows that magnitude errors in χ are less than 3% when $a_0 < 0.9$ and that errors in both magnitude and phase are negligible when $a_0 < 0.2$. The restriction on a_0 may be met by reducing d and f, since ρ and μ are materials properties.

II. EXPERIMENT

A. Example

The dependence of L^* upon a cylinder's length and diameter was computed for a specific pickup coil for illustrative purposes. A plot is shown in Fig. 3. The normalization of the variables applies to this particular coil only.

The room temperature susceptibility of a sample of AISI-type 316 stainless steel was measured using this coil.

TABLE I. Longitudinal demagnetization factors D for cylinders as a function of the ratio of length to diameter l/d [after Crabtree (Ref. 8)].

l/d	D	l/d	D	l/d	D
0.0	1.000	2.0	0.181	4.0	0.0978
0.1	0.796	2.1	0.174	4.1	0.0956
0.2	0.680	2.2	0.167	4.2	0.0935
0.3	0.594	2.3	0.161	4.3	0.0914
0.4	0.528	2.4	0.155	4.4	0.0895
0.5	0.474	2.5	0.149	4.5	0.0876
0.6	0.430	2.6	0.144	4.6	0.0858
0.7	0.393	2.7	0.140	4.7	0.0841
0.8	0.361	2.8	0.135	4.8	0.0824
0.9	0.334	2.9	0.131	4.9	0.0808
1.0	0.311	3.0	0.127	5.0	0.0793
1.1	0.291	3.1	0.123	5.5	0.0723
1.2	0.273	3.2	0.120	6.0	0.0666
1.3	0.257	3.3	0.116	6.5	0.0616
1.4	0.242	3.4	0.113	7.0	0.0573
1.5	0.230	3.5	0.110	7.5	0.0536
1.6	0.218	3.6	0.107	8.0	0.0503
1.7	0.207	3.7	0.105	8.5	0.0473
1.8	0.198	3.8	0.102	9.0	0.0447
1.9	0.189	3.9	0.100	10.0	0.0403

The result was verified with a SQUID susceptometer. A cylinder of length 10.2 cm and diameter 1.2 cm was prepared. $2L^*$ was computed to be 14 μH for the combination of cylinder and pickup coil. An ac field of 570-A/m rms at 100 Hz was applied. χ was found to be 2.9×10^{-3} (SI). This approximates the smallest values of susceptibility which can be accurately measured with the present apparatus. A small piece of steel was obtained from the same rod stock for SQUID measurement. Using a density of 7.958 g/cm^3,[11] χ was 3.0×10^{-3}, in good agreement with the mutual inductance method. Taking a resistivity of 75 $\mu\Omega$ cm,[12] and applying the eddy-current criterion of Sec. I D, a_0 is found to be 0.195, which is very acceptable.

B. Coil compensation

As mentioned in Sec. II A, the background emf of the pickup coil with no sample present needs to be subtracted from the measured emf with the sample in place. This may be done by the various methods shown in Table II. Methods (4), (5), and (6) seem to be satisfactory; method (6), illustrated in Fig. 1, is the least expensive and most straightforward.

III. OTHER CALIBRATION METHODS

A. Change in self-inductance

A variation of the method described in Sec. I A is that of relating the change in self-inductance of a pickup coil, after a specimen is inserted, to the magnetic susceptibility of the specimen. This technique may be used with a commercial inductance meter equipped with a convenient zero offset. Such devices may be thought of as applying a sinusoidal current to a test coil and measuring the voltage across it. The following equations apply:

$$v_s = -L_s \frac{dI}{dt}, \tag{16}$$

where L_s is the self-inductance,

$$I = I_0 \cos 2\pi f t, \tag{17}$$

$$v_{s,\mathrm{rms}} = 2\pi f I_{\mathrm{rms}} L_s. \tag{18}$$

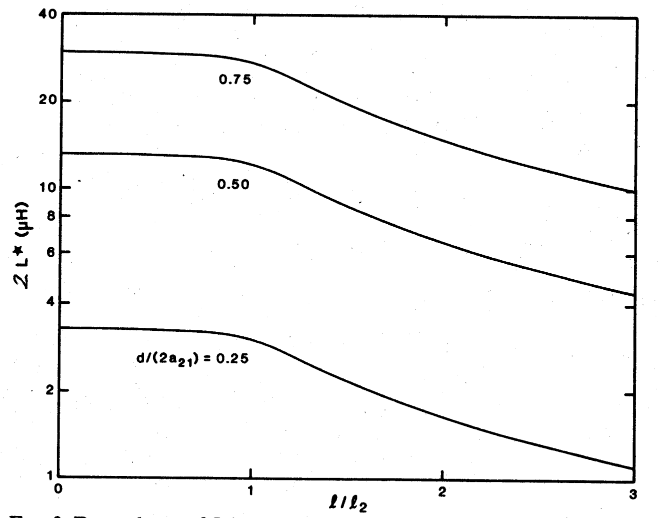

FIG. 3. Dependence of L^* upon cylinder length and diameter. The pickup coil parameters are: $a_{21} = 11.9$ mm, $a_{22} = 13.3$ mm, $l_2 = 127$ mm, $N_2 = 6042$, $z_d = 0$.

TABLE II. Various methods of pickup coil compensation.

	Method of coil compensation	Remarks
(1)	Subtract dc output of lock-in amplifier for empty pickup coil	Simple; poor resolution because $\Delta v/v$ is small
(2)	Null dc output of lock-in amplifier using internal offset	Simple; offset works only over limited range
(3)	Subtract ac signal of pickup coil using digital recording oscilloscope	Any phase shift caused by eddy currents gives errors
(4)	Subtract rms level of ac signal of pickup coil using digital oscilloscope	Digital processing and noise filtering needed
(5)	Subtract an ac waveform equal to that of empty pickup coil, using a differential amplifier, before input to lock-in amplifier	Need stable phase-lock phase shifter with variable gain; eddy-current phase shifts give errors
(6)	Series matched, counterwound "bucking" coil to null sample pickup coil	Both coils must be exposed to same ac field

From Eq. (6), the sample contribution to the voltage is

$$v_{\rm rms} = L^{*} l \chi 2\pi f H_{\rm rms}. \tag{19}$$

The field $H_{\rm rms}$ is proportional to the current in the coil

$$H_{\rm rms} = c I_{\rm rms}, \tag{20}$$

where c is a constant of dimension m^{-1}. The total voltage is

$$v_{s,\rm rms} + v_{\rm rms} = 2\pi f I_{\rm rms}(L_s + cL^{*} l\chi)$$
$$\equiv 2\pi f I_{\rm rms}(L_s + \Delta L). \tag{21}$$

The volume susceptibility is obtained from the increase in inductance ΔL when the specimen is inserted in the coil

$$\chi = \frac{\Delta L}{cL^{*} l}. \tag{22}$$

This method, relying on the coil for both field and pickup, is less sensitive than the method in Sec. I A.

B. Calibration with standards

It is often desirable to calibrate susceptometers experimentally, which requires the use of standards. The resulting calibration is strictly valid only for specimens of the same size and shape as the standard used.

1. Materials with known susceptibility

The NBS Office of Standard Reference Materials sells four susceptibility standards: Al, Pt, Pd, and MnF_2, with χ ranging from 2×10^{-5} to 6×10^{-3} (SI) at room temperature.[13] They are available in various forms.

2. Soft ferromagnets with "infinite" internal susceptibility and known demagnetization factor

An example is a spherical sample of Gd near its Curie temperature. Some provision for cooling is needed to experimentally obtain the maximum pickup voltage.

The susceptibility χ is generally proportional to the pickup coil voltage v, and inversely proportional to the sample volume V, the ac field H, and its frequency f:

$$\chi = \alpha v / V f H. \tag{23}$$

The empirical calibration constant α is obtained by measuring the pickup coil voltage v_0 for a sphere of volume V_0 and demagnetization factor 1/3 in a field H_0 of frequency f_0. Assuming an infinite $\chi_{\rm int}$, Eq. (13) yields

$$1 - D\chi_0 = 0, \tag{24}$$

whence the measured susceptibility $\chi_0 = 3$. Thus,

$$\alpha = 3V_0 f_0 H_0 / v_0. \tag{25}$$

3. Superconducting materials with known demagnetization factor

An example is a spherical sample of Nb below its critical temperature. The ability to cool to cryogenic temperatures is required. The perfect diamagnetism ($\chi_{\rm int} = -1$) of superconductors arises from the Meissner effect. From Eq. (13),

$$\chi_0 = D\chi_0 - 1, \tag{26}$$

and for the sphere, $\chi_0 = -3/2$, or 3/2 but 180° out of phase. Equation (23) applies with

$$\alpha = 3V_0 f_0 H_0 / 2v_0. \tag{27}$$

Similar calibrations may be done, of course, with cylindrical standards using the approximate demagnetization factors in Table I. Methods of preparing spheres are described in Ref. 14.

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