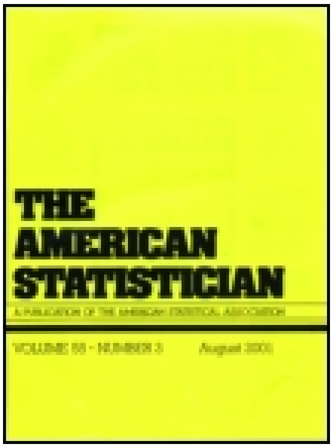

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An Application of the Residue Calculus: The Distribution of the Sum of Nonhomogeneous Gamma Variates

Charles HAGWOOD

The calculus of residues is one of the many beautiful tools that comes out of the field of complex variables. The calculus of residues is applied, together with the inversion formula for characteristic functions, to compute the non-gamma probability density function for the sum of gamma variates with different shape parameters. The distribution of the sum of gamma variates is needed in problems in statistical inference, as well as stochastic processes. This derivation seems more elegant than previous methods for deriving the density function of such a sum. Furthermore, the numerical computation is straightforward, especially in any symbolic computer language.

KEY WORDS: Calculus of residues; Convolution of gamma variates; Gamma distribution; Quadratic form of normals.

1. INTRODUCTION

A random variable with a gamma (λ, r) distribution has probability density function

$$f(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x} \quad x > 0, \tag{1}$$

where r is called the shape parameter and λ is called the scale parameter. Let $Y_1, \ldots, Y_n$ be independent gamma random variables with shape and scale parameters, $(r_1, \lambda_1), (r_2, \lambda_2), \ldots, (r_n, \lambda_n)$, respectively, and let

$$S_n = Y_1 + \cdots + Y_n.$$

It is assumed that each r_i is a positive integer. Several important statistics are distributed like S_n. For example, quadratic forms in normal random variables, such as the $-\log\Lambda$, where Λ is the Wilk's Λ. Robbins and Pitman (1949), Kabe (1962), Gupta and Richards (1979), and others sought the distribution of Wilk's Λ, each using their special technique. A second application occurs as the arrival process of a point process where the inter-arrival times are nonhomogeneous gamma; Sim (1990, 1992) studied such applications. Another application occurs in analysis of baseball home run statistics under the Poisson sampling model. The posterior home run rates will be nonhomogeneous gamma and the average rate over career will be distributed like S_n; see Albert (1992). Also, Johnson, Kotz, and Balakrishnan (1994) devoted a section to the distribution of the convolution of gamma distributions.

When $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \lambda$, the probability density of S_n, $f_n(x)$, is gamma with shape parameter, $r_1 + \cdots + r_n$ and scale parameter, λ. This is easily derived by considering the characteristic function of S_n. The distribution of S_n when the λ_i differ is not gamma and is more difficult to determine. Several authors have derived ways to compute the distribution of S_n using primarily brute-force integration. Coelho (1998) performed the n-fold convolution integration, and Kabe (1962) performed integration in the inversion formula. Sims (1992) gave a formula which he claimed follows via induction. Also, see the work by Waller, Turnbull, and Hardin (1995), Gil-Pelaez (1951), and Imhof (1961).

Kabe's formula is in terms of a generalized multiple hypergeometric function, Exon (1976), sometimes called a Lauicella function

$$f_n(x) = \left[\prod_{i=1}^{n} \lambda_i^{r_i} / (2^{\sum r_i} \Gamma(\textstyle\sum r_i)\right] e^{-r_1 x/2} x^{\sum_{i=1}^{n} r_i - 1} \tag{2}$$

$$\times F_1^{(n-1)} \Big[r_1, \ldots, r_n; \textstyle\sum r_i; \frac{1}{2}(\lambda_1 - \lambda_2)x, \frac{1}{2}(\lambda_1 - \lambda_3)x, \ldots, \frac{1}{2}(\lambda_1 - \lambda_n)x \Big] \tag{3}$$

where the hypergeometric function, $F_1^{(n-1)}$, is given by

$$\sum_{q_2=0}^{\infty} \cdots \sum_{q_n=0}^{\infty} \frac{\prod_{i=2}^{n} (r_i)_{q_i}}{(\sum r_i)_{q_2+\cdots+q_n}} \frac{(\frac{1}{2}(\lambda_1 - \lambda_2)x)^{q_2}}{q_2!} \cdots \frac{(\frac{1}{2}(\lambda_1 - \lambda_n)x)^{q_n}}{q_n!}. \tag{4}$$

Unfortunately, the multiple hypergeometric is not included as a built-in function in most software. The expression Sim (1992) gave involves an infinite series expansion

$$f_n(x) = \frac{1}{\Gamma(a_n)} \Big(\prod_{i=1}^{n} \lambda_i^{r_i}\Big) x^{a_n - 1} \times \exp(-\lambda_n x) \sum_{r=0}^{\infty} \frac{b_n(r)(a_{n-1})_r}{(a_n)_r r!} [(\lambda_n - \lambda_{n-1})x]^r, \tag{5}$$

where

$$a_k = r_1 + \cdots + r_k$$

$$b_i(r) = \begin{cases} 1 & i = 2 \\ \displaystyle\sum_{j=0}^{r} \frac{b_{i-1}(j)(a_{i-2})_j(-r)_j}{(a_{i-1})_j j!} c_i^j & i = 3, \ldots, n \end{cases}$$

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for $r = 0, 1, 2, \ldots$, and

$$c_i = (\lambda_{i-2} - \lambda_{i-1})/(\lambda_i - \lambda_{i-1}). \quad (6)$$

The method used here involves the inversion formula, but the integration is carried out using the calculus of residues, which provides the power to carry out this integration in a simple manner.

2. RESULTS

The characteristic function of the gamma(λ_j, r_j) probability density is

$$\phi_j(t) = \Big(\frac{\lambda_j}{\lambda_j - it}\Big)^{r_j} \quad -\infty < t < \infty \quad (7)$$

and for $r_j \geq 2$, $\phi_j(t) \in L^1(-\infty, \infty)$. Therefore, its probability density function can be recovered from its characteristic function via the inversion formula. By independence and a change of variables

$$\begin{aligned} f_n(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\prod_{j=1}^{n}\phi_j(t)dt \\ &= \frac{\prod_{j=1}^{n} i^{-r_j}\lambda_j^{r_j}}{2\pi}\int_{-\infty}^{\infty} e^{itx}\prod_{j=1}^{n}(t - i\lambda_j)^{-r_j}dt \quad x > 0 \quad (8) \\ &= \frac{\prod_{j=1}^{n} i^{-r_j}\lambda_j^{r_j}}{2\pi} I_n(x). \quad (9) \end{aligned}$$

The integrand as a function of $t = z$,

$$h_n(z) = e^{izx}\prod_{j=1}^{n}(z - i\lambda_j)^{-r_j} \quad (10)$$

is analytic in the upper half plane, except for poles at $i\lambda_j$ of order r_j, $j = 1, \ldots, n$. Therefore, the residue calculus may be applied to evaluate the integral in (8).

Here is a brief summary of the residue calculus. The Cauchy Goursat Theorem states, for a simple closed curve γ and a function $h(z)$ analytic within γ that is continuous on the boundary of γ

$$\int_\gamma h(z)dz = 0.$$

An extension states, if γ is a contour containing $z_1, z_2, \ldots, z_n$, as well as, disks centered at $z_1, z_2, \ldots, z_n$ and $h(z)$ is analytic in the region between γ and the disks, then

$$\int_\gamma h(z)dz = \sum_{j=1}^{n}\int_{C_{R_j}} h(z)dz, \quad (11)$$

where the C_{R_j}'s are the boundaries of the disks, no matter if $h(z)$ is not analytic at the z_j's.

Let γ be the contour which consists of the segment $-R \leq x \leq R$ on the real axis connected to the semi-circle C_R, in the upper half plane, with center at the origin and of radius R. Make R large enough so that disks centered at the poles $z_j = i\lambda_j$, $j = 1, \ldots, n$ of $h_n(z)$ are enclosed by γ. By the extension of the Cauchy–Goursat Theorem

$$\int_\gamma h_n(z)dz = \int_{-R}^{R} h_n(t)dt + \int_{C_R} h_n(z)dz = \sum_{j=1}^{n}\int_{C_{R_j}} h_n(z)dz. \quad (12)$$

At each $z_j = i\lambda_j$, $h_n(z)$ has a Laurent expansion

$$\begin{aligned} h_n(z) = &\frac{b_{r_j}}{(z - z_j)^{r_j}} + \frac{b_{r_j - 1}}{(z - z_j)^{r_j - 1}} \\ &+ \cdots + \frac{b_1}{(z - z_j)^1} + \sum_{k=0}^{\infty} a_n(z - z_j)^k, \quad (13) \end{aligned}$$

where

$$b_m = \frac{1}{2\pi i}\int_{C_{R_j}} \frac{h_n(z)}{(z - z_j)^{-m+1}}dz \quad (14)$$

$b_1 = \frac{1}{2\pi i}\int_{C_{R_j}} h_n(z)dz$ is called the residue of $h_n(z)$ at z_j, written as Residue(z_j). Therefore, substituting in (12) gives

$$\int_{-R}^{R} h_n(t)dt + \int_{C_R} h_n(z)dz = \sum_{j=1}^{n} 2\pi i \ \text{Residue}(z_j). \quad (15)$$

Therefore evaluating the integral in (8) reduces to showing (i) $h_n(z)$ is such that the integral over C_R goes to zero as R goes to infinity, and (ii) finding the residues. It has already been proved that (i) holds; see Curtis (1978). Therefore, letting $R \to \infty$ in (15)

$$I_n(x) = \int_{-\infty}^{\infty} h_n(t)dt = \sum_{j=1}^{n} 2\pi i \ \text{Residue}(z_j). \quad (16)$$

The residue for the pole of order r_j at z_j is given by

$$D^{(r_j - 1)}\Big[e^{ixz}\prod_{k\neq j}^{n}(z - i\lambda_k)^{-r_k}\Big]/(r_j - 1)!, \quad (17)$$

where $D^{(k)}$ denotes the kth derivative operator, see Churchill (1974). So,

$$I_n(x) = \sum_{j=1}^{n} 2\pi i D^{(r_j - 1)}\Big[e^{ixz}\prod_{k\neq j}^{n}(z - i\lambda_k)^{-r_k}\Big]/(r_j - 1)! \quad (18)$$

Most symbolic languages like Mathematica and Maple (see disclaimer) allow symbolic differentiation and can be used to compute $I_n(x)$ from (17), from which one can evaluate

$$f_n(x) = \frac{\prod_{j=1}^{n} i^{-r_j}\lambda_j^{r_j}}{2\pi} I_n(x) \quad (19)$$

without resorting to a recusive formula. Figure 1 plots the density of a nonhomogeneous gamma, using both the recursive method of Sim and by using symbolic features of Mathematica (see disclaimer) to compute $I_n(x)$.

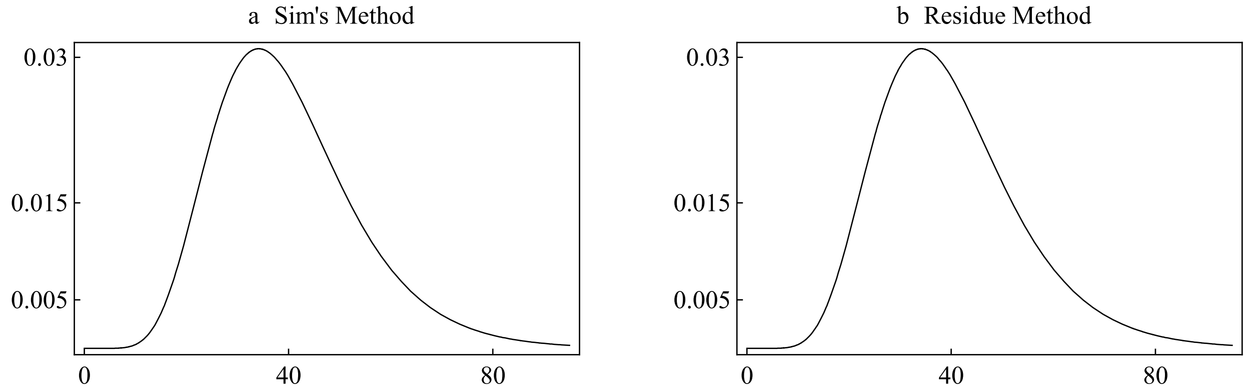

Figure 1. Plots of the nonhomogeneous gamma density with shape $\vec{\lambda} = \{0.1, 0.2, 0.3, 0.4\}$ and scale $\vec{r} = 1, 2, 3, 4$. The infinite series in Sim's formula was truncated at 100.

A closed form formula does exist for the derivative of a product of functions

$$D^{(r_j-1)}\left[e^{ixz}\prod_{k\neq j}^{n}(z-i\lambda_k)^{-r_k}\right]$$
$$= \sum_{m_1,m_2,\cdots,m_n} \frac{(r_j-1)!}{m_1!m_2!\cdots m_n!} D^{m_j} e^{ixz} \prod_{k\neq j}^{n} D^{m_k}(z-i\lambda_k)^{-r_k}, \tag{20}$$

where the derivatives are evaluated at $z = i\lambda_j$ and the multinomial expansion is used in (20)($m_1 + \cdots + m_n = r_j - 1$). These derivatives are given by

$$D^p(e^{ixz}) = (ix)^p e^{ixz} \tag{21}$$
$$D^p(z-i\lambda_k)^{-r_k} = (-r_k)_p(z-i\lambda_k)^{-r_k-p}. \tag{22}$$

Disclaimer: The National Institute of Standards and Technology does not endorse any commerical software product mentioned in this article.

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