Quantum Mechanics of the Electromagnetic Environment in the Single-Junction Coulomb Blockade

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Abstract

We discuss the interaction of a tunneling electron with its electromagnetic environment—when the latter either is in equilibrium or includes an applied microwave field. The environment of an isolated tunnel junction is modeled by a set of harmonic oscillators that are suddenly displaced when an electron tunnels across the junction. We treat these displaced oscillators quantum mechanically, predicting behavior that is very different than that predicted by a semiclassical treatment. In particular, the shape of the zero-bias anomaly caused by the Coulomb blockade (a single-electron charging effect), is found to be strongly dependent on the impedance, $Z(\omega)$, of the leads connected to the junction. Comparison with three recent experiments demonstrates that the quantum mechanical treatment of this model correctly describes the essential

physics in these systems.

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I. INTRODUCTION

Recent advances in microfabrication techniques have allowed the study of tunnel junctions with capacitance so low that the charging energy associated with a single electron can be several meV [1,2]. Under appropriate conditions, this charging energy can cause a suppression of tunneling, called a Coulomb blockade. This suppression has interesting consequences both for normal and superconducting tunnel junctions at low temperatures, including the Coulomb blockade of tunneling [3–6], the Coulomb staircase [7–9], and various oscillatory and dynamic effects such as single-electron tunneling (SET) oscillations and Bloch oscillations [10–15]. Because of the difficulties associated with stray capacitance, the earliest and clearest observations of these effects have been in double- or multi-junction systems. Recently, however, several groups [16–18] have reported the observation of a partial Coulomb blockade in a single junction which lives within an electromagnetic "environment" controlled by parasitic capacitance and inductance in the wires (transmission lines) leading from the measurement apparatus to the junction. The strength and line shape of the blockade appear to be mainly controlled by the (frequency dependent) impedance, $Z(\omega)$, of the electromagnetic environment, with some additional dependence on the impedance of the junction itself. The main purpose of this paper is to enlarge upon earlier discussions [19–21] of the role of quantum fluctuations of the environment on the electrical characteristics of the tunnel junction. We make a quantitative comparison between our theoretical results and the experimental findings of Delsing *et al.* [16], Cleland *et al.* [18] and Gregory [22] and find that the measurements can be correctly accounted for. In particular, the shape of the zero-bias anomaly caused by the Coulomb blockade (a single-electron charging effect), is found to be strongly dependent on the impedance of the leads connected to the junction. In addition, we discuss the effects of quantum charge fluctuations across the tunnel junction which become important when the junction resistance becomes as small as the quantum resistance $R_H = h/e^2$. We also discuss the case of a single junction driven by a microwave field. In this case the electromagnetic environment is not in equilibrium, but rather the mode corresponding to the applied frequency is macroscopically occupied in a coherent state.

II. QUANTUM FLUCTUATIONS AND THE SINGLE-JUNCTION COULOMB BLOCKADE

In the standard semi-classical picture of the Coulomb blockade, the environment is treated classically. The only quantum effect is the ability of an electron to tunnel through the classically-forbidden insulating barrier separating the two sides of the tunnel junction. The origin of the interesting physics of this system is that the charge state of the junction may be viewed as continuously variable (the bias voltage is continuously adjustable) but the discharge process is discrete—only an integer number (one) of electrons crosses the junction during each tunnel event.

We sometimes speak of the charge on the capacitor as being some fraction of an electron. What we really mean by this requires careful discussion. It is best to view the electron fluid in the wires and on the capacitor plates as being dominated by the long-range nature of the Coulomb interaction. This nearly incompressible fluid can be pressed against the capacitor plate by a continuously adjustable force (the bias electric field). This produces a *polarization charge* which increases continuously with applied bias and can correspond to a fraction of an electron. The electron gas is, of course, made up of discrete quanta of charge (*i.e.*, electrons), but that does not alter the fact that the polarization charge is continuously adjustable. Within the standard semi-classical model, the environment is treated classically and hence there is no quantum uncertainty in the polarization charge. We will first examine the semiclassical model and then look at the role of quantum fluctuations.

Let us imagine that a tunneling event "instantaneously" changes the charge state from q to $q - e$. (We will discuss in greater detail below the question of the "duration" of the tunneling event and its effect on the physics.) The change in the potential energy stored in the capacitor is

$$\Delta U = [(q - e)^2 - q^2]/2C_0$$

$$= -eV_0 + e^2/2C_0, \tag{1}$$

where C_0 is the junction capacitance, and $V_0 \equiv q/C_0$ is the initial bias voltage, which due to fluctuations, can be different from the externally applied voltage, V. In the semi-classical model, the only fluctuations are thermal fluctuations. If V_0 exceeds the blockade voltage, $e/2C_0$, ΔU is negative and the tunneling process is "down hill" in energy and hence allowed. Energy conservation is satisfied by the kinetic energy increase of the tunneling electron which ends up above the Fermi level on the other side of the junction.

This simple picture is altered in a variety of ways when the environment is treated quantum-mechanically. One finds that the particle can "tunnel through the blockade," that is, it is possible for an electron to tunnel even when the bias voltage is below $e/2C_0$. There is however, as we shall see, a remnant of the blockade: one finds that the conductance, instead of going to zero below the blockade bias, vanishes as a power law in the limit of small bias. There are several possible ways to explain how the electron is able to tunnel below the classical threshold. One way to view it is to say that the electron is able to tunnel beyond the capacitor plate and part way down the transmission line, thereby taking advantage of the distributed environmental capacitance to reduce the charging energy. An alternative point of view is to say that there is quantum uncertainty in the junction charge q, the junction voltage V_0 fluctuates away from the applied voltage, V, and hence there is uncertainty in the charging energy change ΔU—*i.e.*, the barrier fluctuates. This is really the same as the previous picture however. Imagine that a quantum fluctuation removes some charge from the junction and sends it part way down the transmission line. The tunneling electron then fills in the "hole" on the junction. This is equivalent to having the electron tunnel directly down the transmission line. As we shall see, one should not view the tunneling literally as a single-electron effect, but rather as a collective tunneling of the environmental degrees of freedom (the distributed charge on the transmission line).

In order to tunnel at very low bias, the electron would need to see a large capacitance which requires tunneling a long distance down the line. The action for this process is large and hence the probability is very small, leading to a V^g zero-bias anomaly in dI/dV. The exponent, $g = 2Z/R_H$, of the power law and the action cost for this tunneling are related to the appropriate limit of the transmission line impedance, $Z = Z(\omega = 0)$. This is easily seen from the following uncertainty principle argument. If the junction charge disappears into the environment after a finite discharge time, $\tau_d \equiv C_0 Z$, there is a quantum uncertainty in the blockade energy given roughly by $\Delta E \sim h/C_0 Z$. This is comparable to the blockade energy, $e^2/2C_0$, when Z is on the scale of the quantum impedance, $R_H = h/e^2 \sim 25$ kΩ. For $Z \gg R_H$, we recover the semi-classical limit. Typically, however, Z is much smaller and quantum effects are very important.

In order to treat the environment quantum-mechanically, we have to take a radically different point of view of the Coulomb blockade energy and the tunneling process. We must view the tunneling as an inelastic process that unavoidably "shakes up" the quantum oscillation modes of the environment. That is, the tunneling electron launches a wave which travels down the transmission line. Because this wave disturbance moves with time, it can *not* be an eigenstate of the quantum Hamiltonian and hence *must* be a superposition of states of different energies. Detailed consideration of this energy distribution (excitation spectrum) is crucial to a proper analysis of the current-voltage (I-V) characteristics of the junction.

III. SINGLE-OSCILLATOR "PEDAGOGICAL" MODEL OF THE ENVIRONMENT

We begin our discussion by defining a model Hamiltonian. Our major physical assumption is that the degrees of freedom of the many-body system (electrons plus electromagnetic fields) separate into microscopic single-particle modes and macroscopic collective modes. The total charge on the junction is viewed as arising from two contributions

$$q = ne + q_0, \tag{2}$$

where the "fermionic charge", ne, represents the (integer) number of electrons which have tunneled and q_0 is the "bosonic" collective polarization charge supplied by the transmission line. In a fully microscopic theory, one would have to calculate tunneling matrix elements between many body states that include all of the electronic degrees of freedom as well as the electromagnetic degrees of freedom. Our approximation involves treating the electrons (the single-particle modes) as non-interacting particles that only couple to the electromagnetic modes (macroscopic collective modes) when they transfer a charge from one side of the junction to the other. This approximation allows us to include only single-particle states for the electrons in the tunneling processes. The potential energy in the junction couples the single-particle electron states to the electromagnetic modes, and is

$$U = (ne + q_0)^2/2C_0. \tag{3}$$

As additional electrons tunnel, the "bosonic charge" adjusts itself to keep the net charge on the junction small. This adjustment comes about as the electromagnetic modes are suddenly displaced out of equilibrium due to the fast tunneling process. Below, we discuss in detail the excitation that results from this adjustment, as it determines the details of the Coulomb blockade in these systems.

Degrees of freedom representing bulk and surface plasmons are excluded from the model, because the energy of these modes $\hbar\omega \gg eV$ greatly exceeds typical bias energies. However, these modes are important for the model we use because they allow the separation of degrees of freedom discussed above. After an electron tunnels, it is rapidly screened by the plasmons so that the physical location of the excess charge is independent of the physical location of the tunneled electron. The virtual excitation of these modes does renormalize the tunneling matrix element (by "Franck-Condon factors"), but this merely serves to renormalize what we mean by the "bare" junction impedance.

Furthermore, we neglect other inelastic effects such as coupling to phonons. While electron-phonon scattering will lead to destruction of the phase coherence of the electronic degrees of freedom, no amount of microscopic upheaval in the Fermi sea can change the fact that an extra unit of charge has appeared on the capacitor and can be discharged down the transmission line *only* by collective displacement of those environmental degrees of freedom which we are keeping in the model. We do include the effect of phonon scattering from the electromagnetic modes in so far as that coupling affects the impedance of the transmission lines. We will see in section 4 that the impedance determines how much the electromagnetic modes are excited by the tunneling electron.

To see how the charging energy should be viewed quantum-mechanically, we take a pedagogical model consisting of a tunnel junction capacitor and a *single* inductor L, which represents the entire environment (bias is ignored entirely)

$$H = \frac{L}{2}\dot{q}_0^2 + \frac{1}{2C_0}q_0^2. \tag{4}$$

Putting this into canonical form using the momentum $p_0 = L\dot{q}_0$ yields

$$H = \frac{1}{2L}p_0^2 + \frac{1}{2C_0}q_0^2, \tag{5}$$

which is, of course, a harmonic oscillator of frequency $\omega \equiv (C_0 L)^{-\frac{1}{2}}$. If the oscillator were classical, the minimum energy state would have p_0, $q_0 = 0$. If we consider the sudden tunneling of an electron, the potential energy in the classical picture increases from zero to $e^2/2C_0$ as the charge coordinate is displaced (*cf.* Fig. 1). Quantum mechanics requires us to use a different picture. Initially the system has some probability amplitude for being in charge state q_0

$$\Psi(q_0) = \left(2\pi\left\langle q^2\right\rangle\right)^{-\frac{1}{2}} \exp\left(-q_0^2/4\left\langle q^2\right\rangle\right), \tag{6}$$

where $\langle q^2\rangle = \hbar\omega C_0/2$ is the mean square uncertainty in the polarization charge. In the "sudden approximation," the environment has no time to respond to the tunneling electron, so the wave function remains (at first) unchanged while the Hamiltonian shifts suddenly to

$$H = \frac{1}{2L}p_0^2 + \frac{1}{2C_0}(q_0 - e)^2. \tag{7}$$

It is convenient to make a coordinate transformation $q_0 \longrightarrow q_0 - e$ so that H returns to its old form but the oscillator is now suddenly displaced to

$$\Psi(q_0) \longrightarrow \Psi(q_0 + e). \tag{8}$$

Now we see that the displaced system is not in an eigenstate (stationary state) of H and so will begin to move in time (the oscillator will "ring"). This is the analog of the launching of coherent waves down the transmission line. Expressing the state as a linear combination of excited states

$$\Psi(q_0 + e) = \sum_{n=0}^{\infty} a_n \psi_n(q_0), \tag{9}$$

we see that the final energy of the system is uncertain—but if measured it would necessarily be quantized to an integer multiple of the quantum of energy $\hbar\omega$. While the energy in the harmonic oscillator is uncertain, when we consider the whole system we will find that energy is conserved, the uncertainty here is made up for by differing energies in the electronic quasiparticle states. In fact the final state energy of the oscillator is Poisson distributed, and the probability to be in the n-excitation state is

$$P_n = |a_n|^2 = e^{-\lambda}\lambda^n/n!. \tag{10}$$

where

$$\lambda = \frac{e^2/2C_0}{\hbar\omega}, \tag{11}$$

is the mean number of bosons excited. The quantity $\mathrm{e}^{-\lambda}$ is the square of the overlap between the old ground state and the displaced state. The mean number of bosons excited can be expressed as the ratio of the square of the displacement, $\delta q = e$, to the mean square fluctuations of the charge on the capacitor in the ground state,

$$\lambda = \frac{(\delta q)^2}{4\langle q^2\rangle}. \tag{12}$$

To see the distribution we use second quantization language and write the displaced charge state in terms of a displacement operator acting on the ground state,

$$\Psi(q_0 + e) \leftrightarrow |\Psi(t = 0)\rangle = \mathrm{e}^{-iep_0/\hbar}|0\rangle, \tag{13}$$

where $|\Psi(t = 0)\rangle$ is the state of the oscillator immediately after tunneling and $|0\rangle$ is the ground state. The generator of the displacement

$$p_0 = i\sqrt{\frac{\hbar}{2\omega C_0}}(a^+ - a). \tag{14}$$

is the same momentum operator, conjugate to q_0, that appears in the Hamiltonian, Eq.(7). It is now a simple matter to verify that

$$\begin{aligned} \mathrm{e}^{-iep_0/\hbar}|0\rangle &= \mathrm{e}^{-\lambda/2}\mathrm{e}^{\sqrt{\lambda}a^+}\mathrm{e}^{-\sqrt{\lambda}a}|0\rangle \\ &= \mathrm{e}^{-\lambda/2}\sum_{m=0}^{\infty}\frac{(\sqrt{\lambda})^m}{\sqrt{m!}}|m\rangle. \end{aligned} \tag{15}$$

The first equality follows from the property of operators whose commutator is a c-number,

$$\mathrm{e}^{\mathrm{A}+\mathrm{B}} = \mathrm{e}^{\mathrm{A}}\mathrm{e}^{\mathrm{B}}\mathrm{e}^{-[\mathrm{A},\mathrm{B}]/2}, \tag{16}$$

(*cf.* [23] p. 442). The second equality follows from expanding the exponentials and recognizing that $a^m|0\rangle = \delta_{m,0}$ while $(a^+)^m|0\rangle = (m!)^{\frac{1}{2}}|m\rangle$ (*cf.* [23] p. 436). When projected onto the n-phonon state $|n\rangle$ to find the overlap a_n, the result, Eq.(10) for P_n follows immediately.

The probability distribution, Eq.(10) is illustrated for the case of small and large λ in Fig. 1. For finite λ there is always some probability, $\exp(-\lambda)$, (as in the Mössbauer effect) to leave the system in its ground state—that is, to be able to tunnel elastically with no charging cost. This is a quantum-mechanical effect. In the classical limit ($\hbar \longrightarrow 0, \lambda \gg 1$) the probability distribution becomes a sharply peaked Gaussian centered on the classical value with width determined by random $\bar{n}^{\frac{1}{2}}$ fluctuations in the large number of excited quanta $\bar{n}$: $\langle(\Delta E)^2\rangle^{\frac{1}{2}} \sim \lambda^{\frac{1}{2}}\hbar\omega$, which vanishes as $\hbar^{\frac{1}{2}}$ when $\hbar \longrightarrow 0$. Irrespective of the mean number of excitations, though, the average shake-up energy is

$$\bar{E} = \sum_{n=0}^{\infty} n\hbar\omega P_n = \lambda\hbar\omega = e^2/2C_0 \tag{17}$$

which is precisely the classical value.

From our pedagogical model we see that the mean number of excitations, $\lambda = (e^2/2C_0)/\hbar\omega$, diverges as the frequency of the mode goes to zero. This divergence also

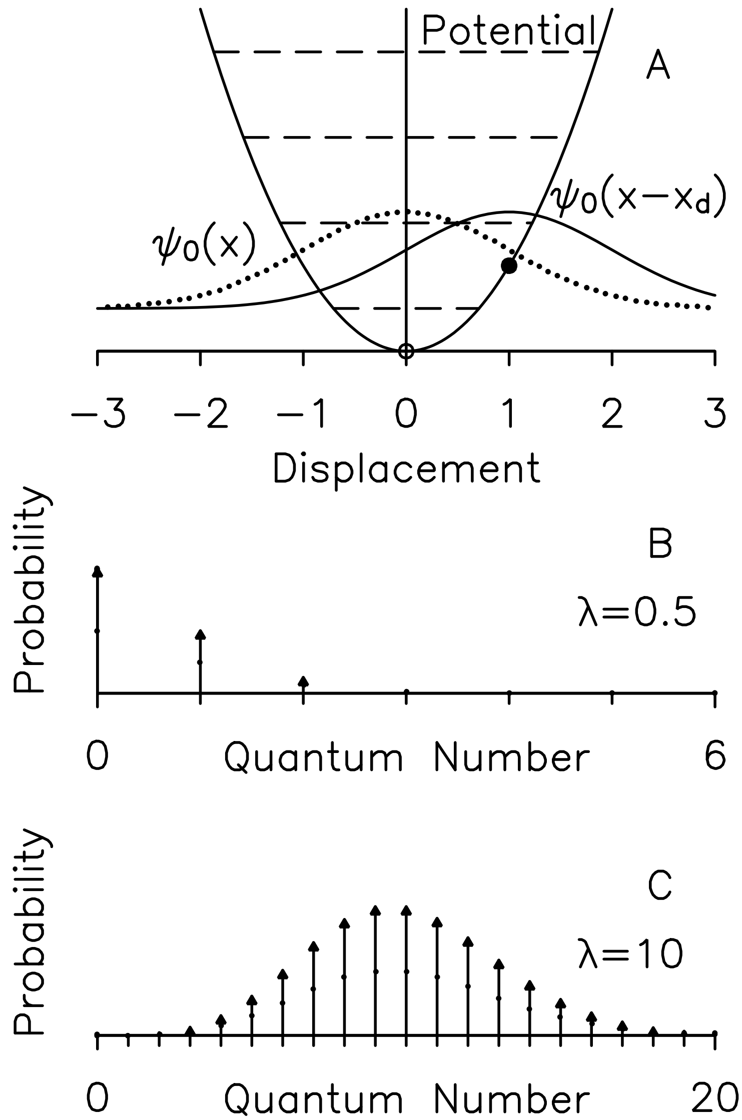

FIG. 1. Suddenly displaced harmonic oscillators. Panel A shows a harmonic potential and the effect of being displaced on classical and quantum particles moving in this potential. When a classical particle in its ground state (open circle) is displaced (closed circle) it still has a well defined energy. When a quantum particle in its ground state (dotted line) is displaced (solid line) its energy is no longer well defined as it is a superposition of all of the eigenstates of the oscillator, each with its own energy (dashed lines). Panels B and C show the probability distribution over the different harmonic oscillator states when an oscillator is slightly and strongly displaced respectively.

occurs for the low-frequency, long-wavelength modes of the transmission line. Because an infinite number of low frequency electromagnetic modes are excited; the probability of being able to tunnel elastically (*i.e.*, at zero bias) is zero. The junction is insulating because the state of the environment is orthogonal to its new displaced ground state. This "orthogonality catastrophe" (*cf.* [24] pp. 744-757) is the source of the power-law zero-bias anomaly in the full transmission-line model, to be discussed in section 4, where we have a continuous distribution of frequencies in the transmission line modes. It is the reason why we find that the conductance is zero at zero bias.

IV. THE TRANSMISSION-LINE MODEL

It is straightforward to extend the above results to the full transmission-line model. We follow the ideas of Caldeira and Leggett [25] and model the low energy electromagnetic degrees of freedom as a set of harmonic oscillators. First we consider ideal transmission lines without dissipation. Then, we generalize to the more realistic case of dissipative lines and lines containing impedance discontinuities which produce reflected waves. The present model [21], contains the same physics as the work of Nazarov [19] and is essentially identical to the model of Devoret *et al.* [20].

After an electron has tunneled, there is an excess charge on the junction capacitor. Classically, the charge on the junction decays as a function of time and a wave of charge is launched down the transmission line. This time dependence is shown in Fig. 2. Classically the wave has a well-defined energy, which is just the charging energy of the junction. Note that for an ideal transmission line, which is dispersionless, the shape of the wave does not change as a function of time. A transmission line can be viewed as the continuum limit of a lumped circuit of inductors and capacitors. Similar to the behavior seen in the pedagogical model discussed in section 3, these inductors and capacitors behave like coupled harmonic oscillators. The wave traveling down the transmission line is a superposition of the normal modes of the transmission line, which have been displaced from their ground state by the

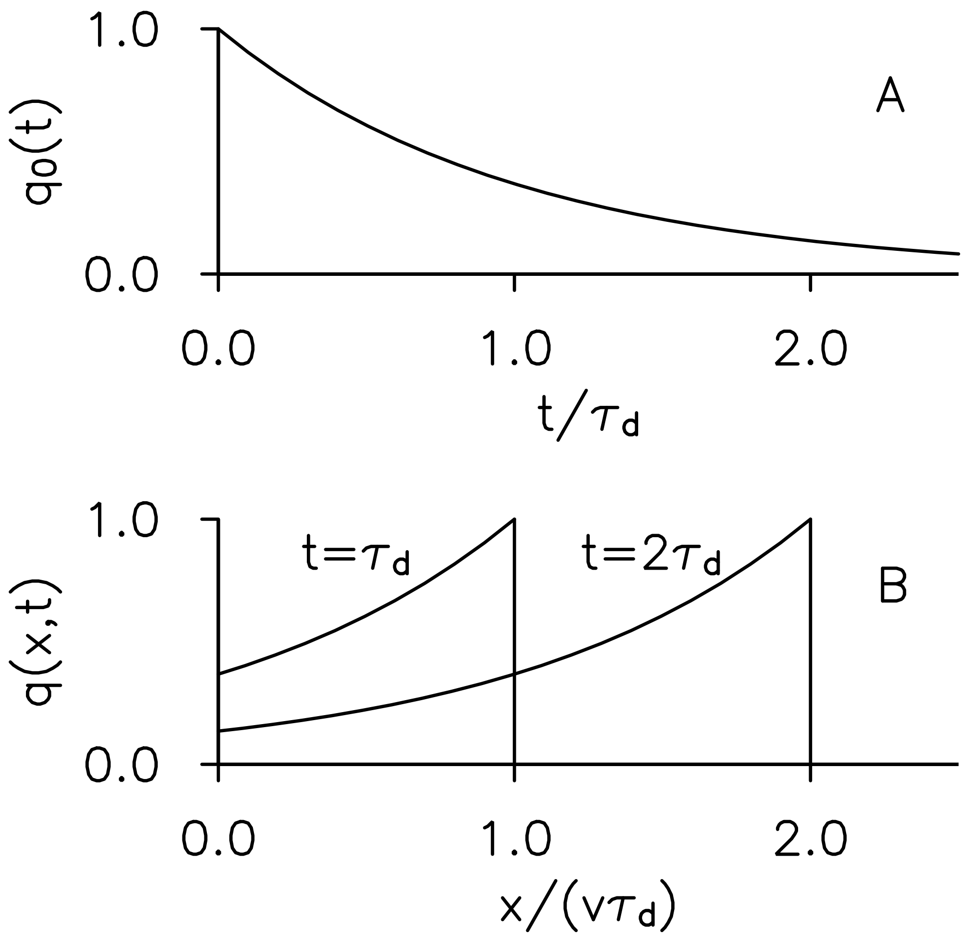

FIG. 2. Classical time dependence of a charge suddenly placed on a junction connected to an ideal transmission line. Panel A shows the decay in time of the charge remaining on the junction and Panel B shows the classical charge wave propagating down the transmission line. The quantity $v = 1/\sqrt{\ell c}$ is the speed of light in the transmission line.

tunneling event.

As we saw in the pedagogical model, a displaced quantum mechanical oscillator does not have a well defined energy as does a classical oscillator. Understanding the behavior of the quantum systems depends on understanding the distribution of energies in the charge wave that has been launched down the transmission line. To find this distribution we need to know what the normal modes of the transmission line are and how much they are displaced when a charge tunnels across the junction. In the following we show how this is done for the case of an ideal transmission line and give the result for a resistive line [21].

The Hamiltonian in Eq.(5) for the pedagogical model is now generalized to

$$H = \frac{1}{2C_0} q_0^2 + \int_0^\infty dx\, \frac{1}{2c} q(x)^2 + \int_0^\infty dx\, \frac{\ell}{2} j(x)^2, \tag{18}$$

where $q(x)$ and $j(x)$ are the excess charge and the current density, respectively; c is the specific capacitance of the transmission line, and ℓ is its specific inductance. The impedance of the ideal transmission line described by this Hamiltonian is a constant, $Z = \sqrt{\ell/c}$, independent of frequency. The excess charge and the current are related by a continuity equation,

$$\frac{\partial}{\partial t} q = -\frac{\partial}{\partial x} j(x), \tag{19}$$

and by an equation of motion

$$\frac{\partial}{\partial t} j(x) = \frac{1}{\ell c} \frac{\partial}{\partial x} q(x). \tag{20}$$

Solving these equations for an ideal transmission line gives dispersionless modes, that is, the frequency of a mode is proportional to its wave vector, $\omega_k = k/\sqrt{\ell c}$. For each mode, the solutions also give the projection onto the charge on the junction. When an electron is suddenly placed on the junction capacitor each of the harmonic oscillators is displaced by an amount equal to the charge transferred, e, times the projection of that mode onto the junction charge,

$$\delta q_k = e \sqrt{\frac{4}{1 + \left(\frac{C_0 k}{c}\right)^2}}. \tag{21}$$

This displacement is a property of the classical equations of motion, and does not depend on the quantum mechanical nature of the oscillators.

The mean excitation of each mode which is given by the ratio of the square of the classical displacement of each oscillator to its quantum mechanical mean square fluctuations, as discussed in Eq.(12). The mean square fluctuations of each mode diverge proportionally to the size of the system, L, as it is taken to infinity,

$$\frac{1}{2c} \langle q_k^2 \rangle = L \frac{\hbar \omega_k}{4}. \tag{22}$$

The probability of exciting each of the modes, which is given by an expression where $< q_k^2 >$ appears in the denominator, will therefore be infinitesimal. But, we will see that even though the probability of exciting a particular mode goes to zero, the probability of not exciting any mode is also zero. For the ideal transmission line we can use the dispersion, $\omega_k = k/\sqrt{\ell c}$, the discharge time, $\tau_d = C_0 Z$, and the coupling constant defined by, $g = 2Z/R_H$, to write the mean excitation of each mode as,

$$\lambda_k = \frac{(\delta q_k)^2}{4\langle\, q_k^2\rangle} = g\frac{1}{1+\omega_k^2\tau_d^2}\frac{1}{\omega_k}\left(\frac{2\pi}{L\sqrt{\ell c}}\right). \tag{23}$$

The coupling constant, g, which was alluded to above, determines the behavior of the blockade at low energies, as is shown below. It emphasizes the quantum mechanical behavior of displaced oscillators by the appearance of the resistance quantum, $R_H = e^2/h$. The classical behavior of the oscillators is emphasized by the Lorentzian factor in Eq.(23) which is proportional to the Fourier transform of the classical time dependence of the charge on the junction capacitor connected to an ideal transmission line. In the limit that the system size goes to infinity, sums over modes are converted into integrals and the inverse of the size of the system becomes the wave vector differential, $L^{-1} \to dk/2\pi$. In this limit the quantity in parentheses in Eq.(23) becomes the frequency differential, $d\omega_k$.

For a displaced quantum mechanical harmonic oscillator the mean number of excitations, λ_k, goes to zero for each individual mode in this system. On the other hand, the sum over all the modes of the mean excitation numbers,

$$\sum_k \lambda_k = g\int_0^\infty d\omega_k \frac{1}{1+\omega_k^2\tau_d^2}\frac{1}{\omega_k} \to \infty, \tag{24}$$

diverges at the lower limit when the system size is taken to infinity and the sum over modes is converted into an integral. Physically, this divergence is due to the excitation of an infinite number of low-energy modes [26]. At the same time, the mean energy in these modes is well behaved,

$$\sum_k \hbar\omega_k\lambda_k = g\int_0^\infty d\omega_k \frac{\hbar}{1+\omega_k^2\tau_d^2} = \frac{e^2}{2C_0}, \tag{25}$$

which is the classical energy.

Given the mean excitation of each mode, λ_k, the distribution of energies excited in the transmission line by a tunneling event, $A(\omega)$, can be written in terms of the spectral density, $a(\omega)$, which is the Fourier transform of the classical time dependence of the charge on the junction. The distribution of energies is given by the sum over all possible sets of excitations weighted by the probability of that set times a delta function of its energy,

$$A(\omega) = 2\pi \sum_{\{n_k\}} \left[\prod_k \mathrm{e}^{-\lambda_k} \frac{\lambda_k^{n_k}}{n_k!} \right] \delta(\omega - \sum_k n_k \omega_k), \tag{26}$$

where we have used the fact that the probability for a mode to be in the n_k-excitation state is given by P_{n_k} in Eq.(10). With a bit of algebra this expression reduces to

$$A(\omega) = \int_{-\infty}^{\infty} dt\, \mathrm{e}^{i\omega t} \exp\left[\frac{e^2}{2C_0} \int_0^{\infty} \frac{d\nu}{2\pi} a(\nu) \frac{(\mathrm{e}^{-i\nu t} - 1)}{\hbar\nu} \right]. \tag{27}$$

For an ideal transmission line the spectral density of transmission line modes displaced by the tunneling charge is a Lorentzian,

$$a(\omega) = \left(\frac{g}{1 + \omega^2 \tau_d^2} \right) (2\pi\hbar) \frac{2C_0}{e^2}. \tag{28}$$

For a more general transmission line Eq.(28) generalizes [21] to

$$a(\omega) = \mathrm{Re}\left[\frac{4}{-i\omega + 1/C_0 Z^*(\omega)} \right] \tag{29}$$

where $Z(\omega)$ is the impedance of the transmission line [27]. Note [28] that $a(\omega)$ can be written as $4C_0 \mathrm{Re} Z_{tot}$, where

$$Z_{tot}^{-1}(\omega) = i\omega C_0 + Z^{-1}(\omega), \tag{30}$$

is the (inverse) total impedance of the tunnel junction capacitance in parallel with the impedance of the environment.

Equation (27) is the excitation spectrum for the electromagnetic modes when an electron suddenly tunnels across the junction. It is valid when the junction resistance, R_0, is large enough that tunneling is rare and each event can be treated independently. The integral over frequency in Eq.(27) does not diverge as the integral in Eq.(24) does because the time-dependent factor goes to zero as the frequency goes to zero and cancels the divergence. In addition, since the long time behavior of the function which is Fourier transformed in Eq.(27) is algebraic rather than exponential there is no delta-function contribution to $A(\omega)$ as there would be if the probability of not exciting any electromagnetic modes were non-zero. This is the manifestation of the orthogonality catastrophe discussed above.

A. Zero Temperature Current

If we assume that the tunneling rate is energy independent in the small interval eV above the Fermi-level, the following simple phase-space arguments give the current at zero temperature. For a junction with an applied voltage, V, an electron can start with an energy, E, in the range 0 to eV. The tunneling event deposits an energy $\hbar\omega$ into the electromagnetic modes with probability density $A(\omega)$. If the final energy of the electron, $E - \hbar\omega$ is below the Fermi level on the other side, then the tunneling is blocked. Integrating over all combinations that contribute to the current gives:

$$I = \frac{1}{eR_0} \int_0^{eV} dE \int_0^{E/\hbar} \frac{d\omega}{2\pi} A(\omega), \tag{31}$$

and the differential conductance thus obeys

$$\frac{dI}{dV} = \frac{1}{R_0} \int_0^{eV/\hbar} \frac{d\omega}{2\pi} A(\omega). \tag{32}$$

It follows that the shake-up excitation spectrum is directly related to the second derivative of current with respect to the voltage,

$$R_0 \frac{d^2 I}{dV^2} = \frac{e}{h} A(eV/\hbar). \tag{33}$$

We use the integral-equation method of Minnhagen [21,29] to solve for this excitation spectrum. This method converts the integral for $A(\omega)$ in Eq.(27) to an integral equation for it. Usually this procedure would be regarded as a step backwards, but it turns out the the resulting integral equation is easy to solve. The conversion proceeds by integrating by parts the right hand side of Eq.(27). Differentiating the complicated exponential with respect to time gives two factors, the original exponential and the Fourier transform of the spectral function. Changing the order of integration converts the right hand side into a convolution of the spectral function, $a(\omega)$, and the energy distribution, $A(\omega)$, resulting in an integral equation

$$A(\omega) = \frac{e^2/2C_0}{\hbar\omega} \int_0^{\omega} \frac{d\nu}{2\pi} a(\nu) A(\omega - \nu). \tag{34}$$

The frequency integration over ν, which formally goes to infinity, is cut off at $\nu = \omega$ because $A(\nu - \omega)$ is zero for $\nu - \omega < 0$ at zero temperature.

The integral equation, Eq.(34), is trivially solved—without any iterative procedure—as the excitation spectrum at a particular frequency only depends on its value at *lower* frequencies (at $T = 0$). It is interesting that the importance of low frequencies emphasizes the long-time behavior. This has to do with the characteristic response time for the displaced oscillators, which is of order $1/\omega$. Because of the Pauli principle for the tunneling electron, only oscillators with response time *longer* than $\hbar/eV$ are relevant. Recall that we consider the limit of large junction resistance, so that there is essentially an infinite time interval before the next tunneling event. In a different interpretation Nazarov [19] states that the tunneling electron must find a final state in an energy interval eV and that it, therefore, has *at most* a "probing" time $\hbar/eV$ to complete the tunneling process. The seemingly contradictory emphasis on long and short times illustrates the ambiguity in the current usage of the concept of "time" in quantum mechanics, well known from the large literature on the traversal time of tunneling [30].

B. Ideal Transmission Lines

To solve the integral equation, Eq.(34), it is necessary to know the asymptotic form of $A(\omega)$ for low frequencies. This form is in turn determined by the impedance through the dependence of the transmission line spectral density $a(\omega)$ on $Z(\omega)$. In the continuum limit of the lumped-circuit model discussed above, the impedance, $Z = \sqrt{\ell/c}$ is constant. Dissipation in ideal transmission lines appears in a somewhat peculiar way; a wave that is launched down such an infinite transmission line never returns, and energy is therefore "lost".

From Eq.(28) we find that the integral equation, Eq.(34), in the low-frequency limit reduces to

$$A(\omega) = \frac{g}{\omega} \int_0^\omega d\nu A(\omega - \nu). \tag{35}$$

This integral equation is solved by

$$A(\omega) = A_0 \omega^{g-1}. \tag{36}$$

The coupling constant, $g = 2Z/R_H$, introduced in Eq.(23) determines the behavior of the excitation spectrum $A(\omega)$ at small frequencies. The normalization constant A_0 is not determined by the integral equation which is homogeneous; it will be discussed further below. Readers familiar with the macroscopic quantum tunneling [25,31,32] or x-ray photoemission [26] literature recognize the characteristic infrared divergence of the shake-up excitation spectrum, which for the present case implies a power-law zero-bias anomaly for the conductance [19–21,33]

$$\frac{dI}{dV} \sim V^g. \tag{37}$$

No matter what the impedance of the transmission line is, the conductance of the junction always goes to zero as the voltage goes to zero. If the impedance is large compared to the resistance quantum, the conductivity is close to zero over a range of voltages approaching the charging energy of the junction. If it is low, on the other hand, the conductivity increases rapidly as the voltage increases.

Starting with the form Eq.(36) the full shake-up excitation spectrum can be calculated numerically from the full integral equation Eq.(34). The results of such calculations are shown in Fig. 3 for a series of different impedances. The excitation spectrum sum rules

$$\int_0^\infty \frac{d\omega}{2\pi} A(\omega) = 1, \tag{38}$$

$$\int_0^\infty \frac{d\omega}{2\pi} \hbar\omega A(\omega) = \frac{e^2}{2C_0}, \tag{39}$$

set the normalization, and are useful for checking the numerical accuracy. They also guarantee that, in the limit of large voltage, the conductance will obey the usual Coulomb offset

$$I(V) = \frac{V - e/2C_0}{R_0}. \tag{40}$$

We can estimate the voltage scale for reaching this limiting result by assuming that the low-frequency form, Eq.(36), holds for *all* frequencies up to a cutoff ω_c. Using the two sum rules, the cutoff frequency (and a normalization constant) can be determined. One finds that

$$\hbar\omega_c = \frac{g+1}{g} \frac{e^2}{2C_0}. \tag{41}$$

Hence the shake-up excitation spectrum for small g, in addition to the divergence at low frequencies, is characterized by a long tail out to frequencies of order $1/g$ times the charging energy. The origin of this Lorentzian tail is the energy uncertainty h/τ_d associated with the rapid discharge of the tunneled electron into the transmission line. The existence of this effect means that for small g the I-V characteristic does not achieve the sum-rule form Eq.(40) until rather large voltages $V \sim e/2C_0 g$; *i.e.* there is an extensive regime over which the I-V curve is weakly nonlinear. For large g, the shake-up excitation spectrum is peaked around the classical energy, and the full Coulomb blockade is recovered (smeared out by quantum fluctuations). This behavior is evident in Fig. 3, where the excitation spectrum $A(\omega)$, the differential conductance, and the current-voltage curves are displayed for values of g in the range 10^{-2}–10^2.

C. Resistive Transmission Lines

It is difficult to make a transmission line whose impedance differs significantly from that of free space, $Z \sim 377\Omega$, which is much smaller than the quantum resistance and

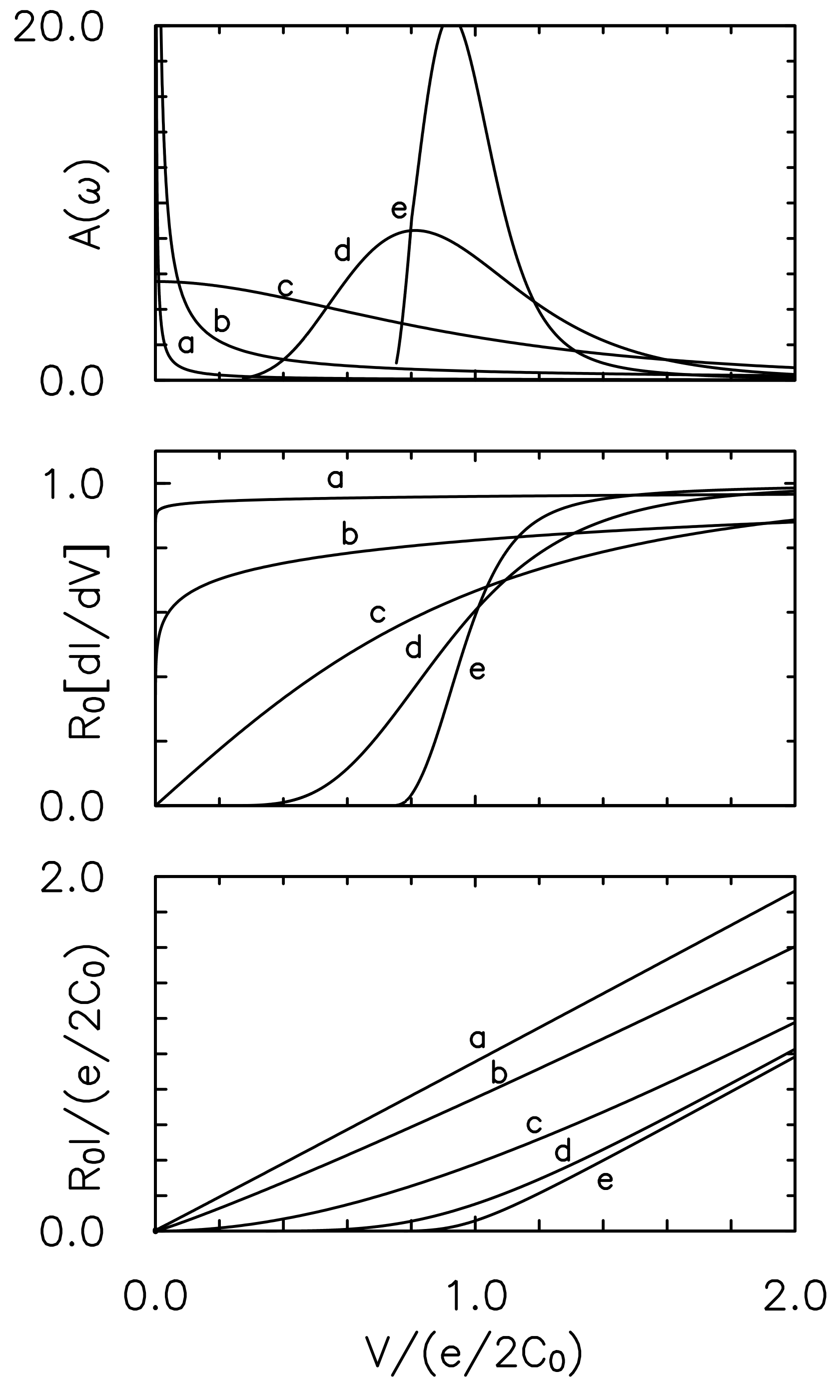

FIG. 3. Ideal transmission lines. The excitation spectrum $A(\omega)$, the differential conductivity dI/dV, and the current I are plotted for a series of transmission lines with coupling constants, $g = 2Z/R_H$, (a) $g = 0.01$, (b) $g = 0.1$, (c) $g = 1.0$, (d) $g = 10$, and (e) $g = 100$. The curves are scaled by the differential resistance of the junction in absence of charging effects, R_0, and by the voltage characteristic of the charging energy $e/2C_0$.

hence gives small coupling constants. One way around this difficulty is to make highly disordered, resistive leads. Thus, it is interesting to consider the case of a semi-infinite, dissipative transmission line. In the spirit of Caldeira and Leggett [25], we include extra bosonic degrees of freedom to model the dissipation (each resistor element in a lumped circuit model can be modeled by its own semi-infinite dissipationless line). One interesting aspect of the highly disordered transmission lines involves localization and Coulomb interaction effects. Since it is readily possible to fabricate narrow wires with resistances well in excess of the quantum resistance, strong localization can manifest itself at low enough temperatures. The strong back-scattering interference effects of localization would produce a conductivity which initially increases with frequency. The Caldeira-Leggett formalism captures some of this physics, to the extent that it is reflected in the linear response properties (the frequency dependence of the impedance) of the line. The Caldeira-Leggett oscillator formalism obtains the full non-linear response by essentially exponentiating the linear response. The validity of this in the nearly localized or strongly Coulomb correlated regimes is perhaps less clear. It would be interesting to see experiments which use the Coulomb blockade as a probe of localization effects.

To find the excitation spectrum for a resistive transmission line, we use the generalization of the results for the spectral density, Eq.(29), and the impedance of such a line,

$$Z(\omega) = \sqrt{\frac{l + r/i\omega}{c}}, \tag{42}$$

where r is the resistivity of the transmission line. The impedance of the semi-infinite resistive transmission line diverges as $\omega^{-\frac{1}{2}}$ in the low-frequency limit, where

$$a(\omega) = \frac{1}{\sqrt{\omega}}\left[\sqrt{2}\frac{1}{R_H}\sqrt{\frac{r}{c}}\right]. \tag{43}$$

It turns out that a solution to the integral equation can be found in this case also [34]

$$A(\omega) = A_0 \frac{\mathrm{e}^{-\omega_o/\omega}}{\omega^{3/2}} \tag{44}$$

where A_0 is a normalization constant which is undetermined by the homogeneous integral equation, and

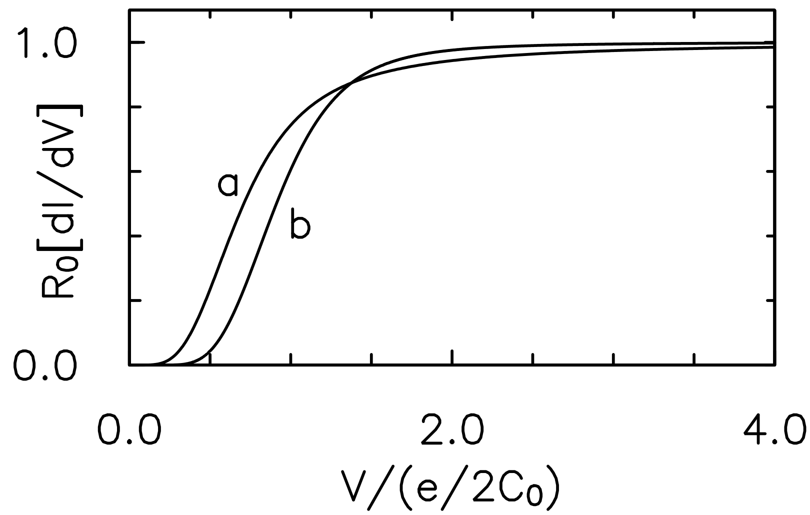

FIG. 4. Comparison of the differential conductivity for (b) an ideal transmission line with $g = 10$ and (a) a resistive transmission line with $r = 8000\ \Omega/\mu$m, $\ell = 600$ fH/μm, and $c = 0.022$ fF/μm.

$$\omega_o = 2\pi \frac{1}{R_H^2} \frac{r}{c}. \tag{45}$$

This frequency sets a voltage scale, $V_o = \hbar\omega_o/e = (e/c)(r/R_H)$. There is a strong zero bias anomaly, but no power-law behavior. At a frequency $\sim r/c$ there is a crossover and again one finds long "tails" at large voltages before the asymptotic value of unity for the normalized differential conductance is reached, as illustrated in Fig. 4

A model more readily comparable with experiment [21] involves a (possibly) resistive transmission line of finite length d terminated by the impedance $Z_t(\omega)$ that could, for instance, be due to wide metallic contacts. The impedance is

$$Z(\omega) = Z_0(\omega) \frac{1 + a\mathrm{e}^{-2if(\omega)d}}{1 - a\mathrm{e}^{-2if(\omega)d}}, \tag{46}$$

where $Z_0(\omega)$ is the impedance of a semi-infinite transmission with the same characteristics of the finite section, and is given by Eq.(42). The quantities a and f are given by

$$a = \frac{Z_t(\omega) - Z_0(\omega)}{Z_t(\omega) + Z_0(\omega)}, \tag{47}$$

and

$$f(\omega) = \sqrt{lc\omega^2 - i\omega rc}. \tag{48}$$

The zero-frequency limit $Z(0) = rd + Z_t(0)$ simply gives the finite total resistance seen from the junction, and determines the power-law behavior of the differential conductance at small voltages. As we discuss in Sec. 6, it is interesting to consider the possibility of resonances due to the geometrical structure of the leads. The resonances would manifest themselves as weak oscillations in the I-V characteristics.

It is interesting to ask if there is an orthogonality catastrophe for a junction fed by a resistor R of finite length. Naively, this finite length cuts off the infra-red divergence. However this is *not* the case. Since we are dealing with quantum mechanics we must have a Hamiltonian for the resistor. Lacking the actual microscopic Hamiltonian we use the Leggett picture [25] and model the resistor as an infinite collection of oscillators which produce ohmic dissipation. A concrete realization of such a set of oscillators is an infinite transmission line of impedance R. A voltage pulse applied to one end launches a wave which never returns and hence irreversibly dissipates energy just as the resistor does. This boson approximation for the resistor degrees of freedom has been shown to be exact within the framework of the linear response approximation and argued to be the correct result in general [25]. Hence, because of the irreversibility, we fully expect the orthogonality catastrophe in a finite resistor [35].

D. Generalization to Finite Temperatures

At finite temperatures a formal calculation [36] confirms the "obvious" generalization of the zero temperature expression, Eq.(27), for the shake-up excitation spectrum. The argument of the exponential becomes,

$$\begin{aligned} &\left(\mathrm{e}^{-i\omega t} - 1\right) \longrightarrow \\ &\{\, [n_B(\omega) + 1] \left(\mathrm{e}^{-i\omega t} - 1\right) + n_B(\omega) \left(\mathrm{e}^{i\omega t} - 1\right) \} \end{aligned} \tag{49}$$

where $n_B(\omega) = 1/(\exp(\hbar\omega/k_B T) - 1)$ is the Bose factor, and the two terms correspond to stimulated photon emission and photon absorption. Using the generalized excitation spectrum we write the tunneling current from left to right as

$$I_{l\to r} = \frac{1}{eR_0}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}A(\omega)\int_{-\infty}^{\infty}d\epsilon\, n_l(\epsilon)\left[1-n_r(\epsilon-\hbar\omega)\right]. \tag{50}$$

Here $n_l(\epsilon)$ is the probability that the state of energy ϵ on the left side is occupied and $1 - n_r(\epsilon - \hbar\omega)$ is the probability that the state of energy $\epsilon - \hbar\omega$ on the right side is empty. Hence we have taken the Pauli principle into account. Note that energy is conserved in the tunneling process. A similar expression describes tunneling from right to left. We assume that there is a potential difference between the left and right sides of the tunnel junction, each in equilibrium, $n_l(\epsilon) = n_F(\epsilon)$ and $n_r(\epsilon) = n_F(\epsilon - eV)$ with $n_F(\epsilon) = 1/(\exp(\epsilon/k_BT)+1)$. By using the mathematical identity

$$\begin{aligned}-n_F(\epsilon-\epsilon_1)n_F(\epsilon-\epsilon_2) = n_B(\epsilon_1-\epsilon_2)n_F(\epsilon-\epsilon_1) +&\\ n_B(\epsilon_2-\epsilon_1)n_F(\epsilon-\epsilon_2)&\end{aligned} \tag{51}$$

we can perform the integration over ϵ to get for the total current

$$\begin{aligned}I = \frac{1}{eR_0}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}A(\omega)[(\hbar\omega - eV)n_B(\hbar\omega - eV) -&\\ (\hbar\omega + eV)n_B(\hbar\omega + eV)].&\end{aligned} \tag{52}$$

In the zero temperature limit the Bose function is a simple step function

$$n_B(\hbar\omega + eV) \longrightarrow -\Theta(eV - \hbar\omega) \tag{53}$$

and we recover the earlier results.

At finite temperatures Minnhagen's integral-equation trick does not work, because the excitation spectrum $A(\omega)$ at a particular frequency would no longer depend only on lower frequencies. Instead we use fast Fourier transform techniques which are quite efficient. With typically 250 000 grid points each I-V curve requires about three minutes of CPU-time on a workstation. Figure 5 shows a typical example of the temperature dependence of the differential conductance.

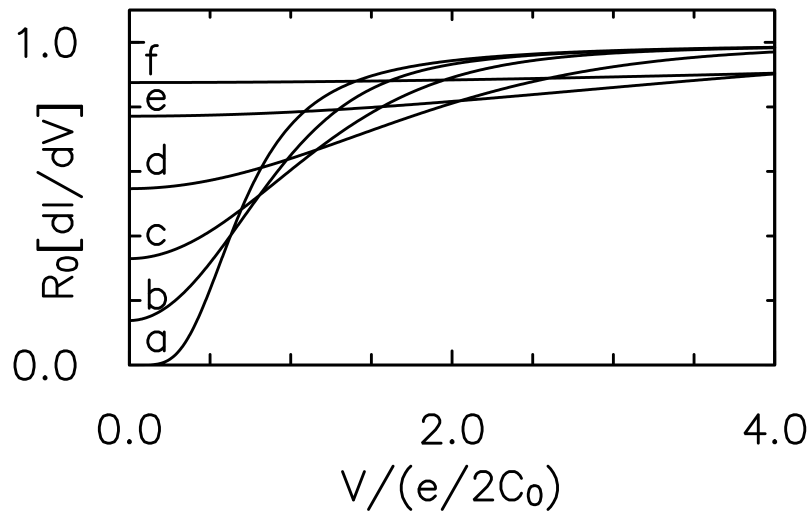

FIG. 5. Temperature dependence of the differential conductance for a resistive transmission line as in Fig. 4. The temperatures are (a) 0K, (b) 0.5K, (c) 1.0K, (d) 2.0K, (e) 5.0K, (f) 10.0K. The charging energy of the junction is 0.4 meV or 4.6 K $\cdot k_B$.

V. BEYOND THE TRANSMISSION LINE MODEL

In addition to the assumptions we discuss above, the separation of degrees of freedom, and the neglect of high energy plasmon modes, there are several other approximations that are implicit in the present model. We assume that the tunneling process is instantaneous so that the electromagnetic modes behave as if they have been suddenly displaced. We also assume that higher order processes in perturbation theory in the tunneling matrix element are not important. This assumption manifests itself in several ways. First, we assume that the tunneling events are far enough apart in time that they do not affect each other; second, we assume that virtual tunneling events do not strongly renormalize these results, or wash them out. We further assume that the electrons in the leads form a good Fermi liquid. Here, we discuss some of these assumptions.

A. Finite Traversal Time

The chief assumption that has been made is that the environment oscillators are displaced suddenly by the “instantaneous” tunneling. We turn now to the question of the duration of the tunneling event. If we use the Landauer-Büttiker picture as a characteristic time scale for the very short but finite traversal time τ_T of the electron, then we see that the sudden approximation breaks down [21,37] for short-wavelength modes k with frequencies $\omega_k \tau_T > 1$. For the highest frequencies, it is more appropriate to use the *adiabatic* approximation in which the displaced oscillator gradually moves to its new ground state with essentially unit probability. Physically this means that as the electron tunnels, the shortest wavelength transmission line modes "see it coming" and begin to transfer charge down the line away from the approaching electron. Given the $\omega_k \tau_T = 1$ dividing line (which is of course not perfectly sharp) between sudden and adiabatic, we see that the adiabatic displacement will affect the transmission line out to the "horizon" distance $\tau_T c$. However, the only significant effect of this is to slightly reduce the mean shake-up energy; *i.e.*, renormalize the effective junction capacitance upwards by the amount of transmission line capacitance distributed in the distance $\tau_T c$. The exponent g of the zero-bias anomaly is unaffected since it is independent of C_0. The "offset voltage" $e/2C_0$ as well as the range, Eq.(41), of the non-linear region, on the other hand, will be reduced. For superconducting junctions the effective tunnel barrier for the "phase particle" can be quite small, $\tau_T c$ can be large, and τ_T can in fact measured [32]. However for normal junctions the oxide barrier height is so large that τ_T is of order 1 fs and $\tau_T c$ probably no more than a few tenths of a micron. While the existence of a single traversal time may be problematical, for these systems the appropriate times are almost certainly very short.

B. Finite Junction Resistance

From the results of our pedagogical model and the exact solution of the general transmission line problem, we see that there is an infrared divergence associated with the excitation of an infinite number of low energy quanta. As a result, a displaced state of the oscillators is orthogonal to the ground state. This "orthogonality catastrophe" means that the probability to tunnel elastically (with no shake-up) is zero, and there is thus a singular zero-bias anomaly in which dI/dV vanishes as a power law at low bias.

The solution we have obtained is the exact result to leading order in an expansion in $1/R_0$, the "bare" conductance of the junction. For finite values of R_0 we must consider the effects of interference and correlation among multiple tunneling events [38,39]. Of particular interest is the question of the effect of a finite value of R_0 on the zero-bias anomaly.

The excitation spectrum $A(\omega)$ of the electro-magnetic modes given by Eq.(27) is exact. As discussed in section 7.2 below, this result is related to the fact that a lowest-order cumulant expansion of $\langle 0|\exp(iep_0(t)\exp(-iep_0(0)|0\rangle$, which only depends on the correlation function $\langle 0|p_0(t)p_0(s)|0\rangle$, is exact for harmonic oscillators. In order to treat higher order effects in $1/R_0$ a lowest-order cumulant expansion also in the fermion operators was done in Ref. [36] using a path integral formulation. This is an *approximation*, which allows for multiple correlated hops but treats the microscopic electron tunnel events using the equilibrium fermion Green's functions. Viewing the path integral as a statistical mechanics problem, tunnel events appear as positive and negative "charges" which interact logarithmically and are therefore correlated. Charges of different sign are attracted to each other and hence when few charges are present the most likely event after an electron tunnels forward across the junction (positive charge) is a tunneling backwards (negative charge), which tends to suppress the current. The chemical potential for the charges is related to the junction resistance, R_0. If R_0 is low the energy cost for creating charges (tunnel events) is small, and for sufficiently low junction resistance—with many charges present—screening effects may lead to an unbinding of the positive and negative charges and to a crossover or transition from a blockade state to a conducting state.

If we ignore the extra factors of the fermion Green's functions, the equivalent statistical mechanics problem is closely related [40] to the problem of a superconducting phase "particle" moving in a frictional medium [41–43]. It is known that there is a phase transition (at zero temperature) which, when translated into the present problem, would imply a finite conductance at zero bias for $g < 1$ and a blockade for $g > 1$. The transition is controlled however by the coupling constant g and *not* the value of the junction resistance R_0 relative to R_H. If this analogy holds, there is presumably a temperature below which the difference between this new conducting state and the blockade state is significant and above which it is not. While the existence of a transition to a renormalized conducting state depends on the coupling constant, g, rather than the junction resistance, this crossover temperature must depend on the junction resistance. Clearly this is an interesting question that deserves further study.

In recent theoretical work on ground state properties (as opposed to the conductance) of normal tunnel junctions indications of a transition from a blockade state to a conducting state controlled either by the junction resistance [39,44,45] or the resistance of the external circuit [38] have been found.

For small enough R_0 we presumably cannot make the lowest-order cumulant expansion for the fermions which uses equilibrium fermion Green's functions. Particles tunnel back and forth quickly and, as in the Kondo problem, the single-particle occupation numbers "remember" the history of tunnel events. It is possible that there is a crossover (as in the Kondo problem) to a "Fermi-liquid-like" state with renormalized, but finite conductance.

There is an important distinction between the *continuous* charge fluctuations in the transmission line and the *discrete* fluctuations (in units of e) across the junction. In a Gaussian fluctuation or "spin-wave" approximation [36,46,47] this distinction is ignored, however one obtains a tractable model in which the essential effect is a simple renormalization of the effective impedance [36],

$$Z^{-1}(\omega) \longrightarrow Z^{-1}(\omega) + R_0^{-1}, \tag{54}$$

due to the addition of the fluctuations across the junction and in the leads. This approximation correctly weakens the blockade for small R_0/R_H, but presumably cannot capture all of the physics discussed above, associated with the discreteness of the charge.

C. Electron-Hole Pair Excitations

In a transmission line there are both electromagnetic (photon-like) boson modes and electron-hole pairs which behave as bosons. Since the electron-hole pair excitations create a charge disturbance, these modes are coupled. In the present model we have assumed that the important modes for low-energy losses are the combined modes that are derived in some adiabatic sense from the electromagnetic modes. We ignored the rest of the electron-hole pair derived modes. In contrast Ueda and Kurihara [48] have suggested that these modes *also* give rise to an infrared divergence and, therefore, should be important. We strongly doubt that this conclusion is correct.

Although Ueda and Kurihara's theory is formally analogous to that used for the x-ray edge problem, there are important differences between that problem and the tunneling problem. In the x-ray edge problem a high energy interaction suddenly creates a localized, immobile core-hole. In the tunneling problem a low energy tunneling event suddenly creates a charge disturbance that is neither localized, nor immobile on the *relevant length scale* for electron-hole pairs. Since the charge disturbance is spread out spatially, the potential that scatters the electron-hole pairs, $V_{kk'}$, is appreciable only for $q = k - k'$ small. The restriction to small q substantially reduces the phase space available for exciting electron-hole pairs.

Without a spatially localized charge disturbance (like the core hole in the x-ray edge problem) Fermi liquid theory guarantees that there will not be any singularities due to electron-hole pairs. If the potential is not localized on an atomic scale, the potential only has low wave vector components, but in a Fermi liquid all of the low wave vector spectral weight is in the plasmons and not the electron-hole pairs. In addition, the charge disturbance is extremely mobile as it moves down the transmission line at the speed of light. This means that the scattering potential has to be time-dependent on the scale of electron-hole pair energies and cannot be treated as static as it is for a core-hole excitation. Hence the effect of the electron-hole pair modes should be much smaller than the effect of the electromagnetic modes contained in the present model. In particular the excitation spectrum for the electron-hole pair excitations should not be singular at low frequencies as is the excitation spectrum, Eq.(36), of the electromagnetic modes.

VI. COMPARISON WITH EXPERIMENTS

Here we compare the results of the transmission line model with three measurements. We find that the model agrees well with with the size of the zero-bias anomalies measured for isolated single junctions. Quantitative comparisons between theory and experiment are complicated by the difficulty in independently measuring the details of the systems. However, based on the ability to fit most aspects of the experimental data with parameters consistent with the experimentally estimated parameters, we are confident that the present model correctly describes the effect of the electromagnetic environment in these systems.

First we consider the experiment of Delsing *et al.* [16]. Fig. 6 shows measured and calculated conductivities of an isolated junction in a low impedance electromagnetic environment. The Coulomb blockade is almost completely washed out and the theory qualitatively accounts for this. The difference between the two calculated curves illustrates the difficulty in comparing theory with experiment. The dotted curve, which does not agree well at all, was calculated using the parameters estimated in the experimental paper. The solid curve was calculated by adjusting those parameters a little to get a zero bias anomaly of about the correct size. The new parameters are probably within their experimental uncertainties. An obvious difference between the calculated and the measured curves is the long tails seen in the calculated curves. We believe that these tails are not found in the experimental curves just because of the difficulty in knowing experimentally what the asymptotic value of the resistance is.

One of the interesting features of the data is the appearance of small oscillations in dI/dV in the wings of the curve. Nazarov [49] has attributed these oscillations to random features in a universal conductance fluctuation type of model. It is not clear that random fluctuations of this type can explain the rather periodic oscillations in the data. We have considered the

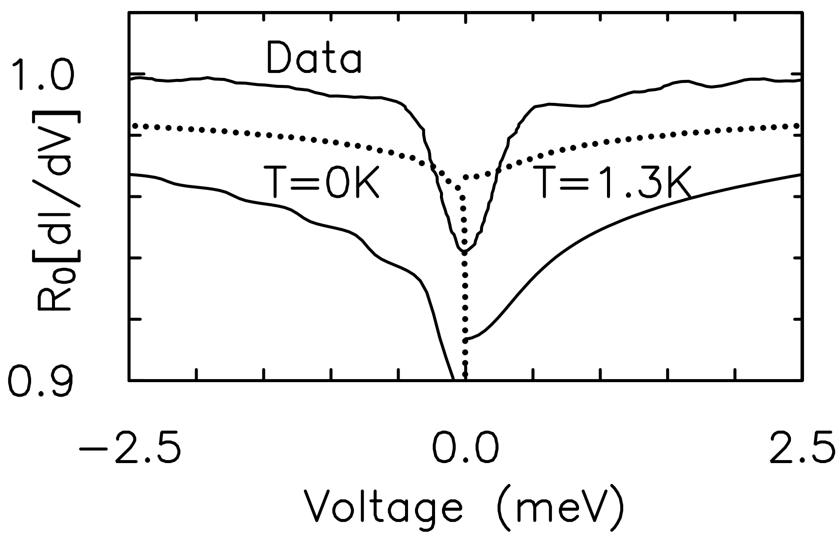

FIG. 6. Comparison of theory and the measurements of Delsing *et al.* [16]. The conductivity of an isolated junction is shown from both measurement (curve labeled data) and calculation. The calculated curves are shown at 0K for negative voltages and 1.3K for positive voltages. The solid curves were calculated using a specific inductance of, $\ell = 700\text{fH}/\mu\text{m}$, and a specific capacitance of $c = 0.01\text{fF}/\mu\text{m}$; the dotted curves were calculated using $\ell = 600\text{fH}/\mu\text{m}$ and $c = 0.1\text{fF}/\mu\text{m}$. The structure in the zero temperature curves is due to structure in the impedance due to a discontinuity in the leads 1.5 mm from the junction. This structure is washed out at higher temperatures. The rest of the parameters for the calculations are discussed in the text.

possibility that they are due to wave reflection from the discontinuity in the structure at the contact pad which is located [50] 1.5 mm away from the junction. The reflected waves produce periodic resonances in $Z(\omega)$ and hence modulation in the conductance, which by suitable choice of parameters can be made qualitatively consistent with the data. However, we find that the oscillations wash out very rapidly with temperature. Also, as the shake-up excitation spectrum $A(\omega = eV) \sim d^2I/dV^2$ is positive definite, the differential conductance is necessarily a monotonic function of voltage in this model, while in the experiment it seems not to be. As the data is not symmetric in voltage, it is conceivable that variations in the electron density of states—assumed constant in the present model—is an important factor. Alternatively, resonant tunneling through localized states in the barrier might play a role. At present the origin of the oscillations are not properly understood.

In Fig. 7. a comparison is made with the experiments of Cleland *et al.* [18], where a single junction is connected by resistive transmission lines to the measuring apparatus and the current source. To compare a large range of experimental conditions it is useful to plot the differential resistance at zero bias under the different conditions. Here we plot the differential resistance at zero bias as a function of the ratio of the temperature to the charging energy of each junction. In the semiclassical model all of the results would fall along the same curve. The deviations of the data—due to quantum fluctuations— from the semiclassical result are well accounted for by the model, in particular at high temperatures. The model also reproduces the largest differences between the different sets of experimental data, which are due to the use of leads with different resistivities. The agreement gets worse as the temperature gets lower; the experimental data saturates while the calculated results continue to increase. The lack of agreement is due to the power-law divergence of the resistivity, $1/V^g$, at zero temperature in the present model. We speculate that including quantum fluctuations across the tunnel junction may account for the saturation. If these are included in the "spin-wave" approximation discussed above, a slightly better agreement with the data follows, but the power-law divergence persists. It is possible that to get saturation it is necessary to use a theory which keeps the discrete nature of the charge fluctuations (in units of e) across the junction.

In an interesting experiment of a different type, Gregory [22] measured the differential conductance between a series of two crossed platinum wires, separated by an adjustable-thickness (frozen) helium film. In the series, the junction resistance is found to vary as would be expected if the distance between the wires varied. He observes a zero-bias anomaly and at larger voltages a quadratic contribution, presumably due to the voltage dependence of the tunneling matrix element, which was subtracted off. The zero-bias anomaly decreases and eventually disappears as the junction resistance approaches from above, a resistance close to the resistance quantum. As discussed in section 5.2, there are indications that this behavior is to be expected.

The comparison of the present model with this experiment is more speculative than it is for the other experiments, mainly because less is known about these systems than the other systems. In particular, the capacitances of the junctions are not independently measured, and the impedance of the environment near the tunneling path is not known. Gregory assumes that the junction capacitances are quite small, $\sim 10^{-18} - 10^{-17}$F, and that the impedance in the neighborhood of the junction is on the order of the resistance quantum. He attributes the differences between the measurements made for different junctions as due to the variation of the junction resistance between the different measurements, assuming that the capacitances of the junctions do not vary. He also assumes that almost the full blockade is observed due to the large impedance he assumes for the immediate environment of the junction.

While the present model cannot accurately treat junctions with resistances close to the resistance quantum, we think that it can explain some aspects of the experimental data. In particular, we assume that the impedance of the environment is just due to the impedance of the platinum wires, which would be close to the impedance of free space, and we assume that the capacitances of the junctions are larger, in the range from 1.6×10^{-17}F to 8×10^{-15}F. In Fig. 8 we show how the the present model might describe these measurements based on these assumptions. The coupling constant is chosen to be twice that of free space, 2.9×10^{-2}.

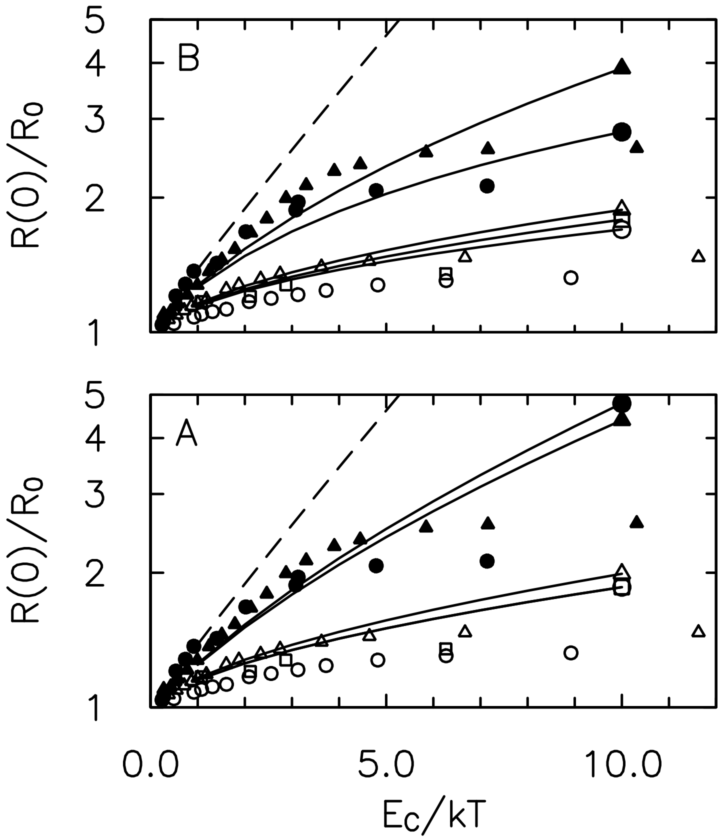

FIG. 7. Comparison of theory and the measurements of Cleland *et al.* [18]. The zero bias resistivity is plotted vs. the charging energy divided by the temperature for a series of temperatures and a set of junctions. The data for each junction are shown using a different symbol. Calculated curves for each junction are labeled by attaching the appropriate symbol to the end of the theoretical curve. Panel A shows the results of calculations that ignore the finite junction resistance; panel B includes the finite junction resistance in the spin-wave approximation (see text). The dotted line gives the semiclassical prediction. The solid symbols are for junctions connected to high resistivity NiCr leads, $r = 30\Omega/\mu$m, and the open symbols are for lower resistivity CuAu leads, $r = 30\Omega/\mu$m. All of the leads were $d = 12000\mu$m long with a specific capacitance of $c = 0.0098$fF$/\mu$m, and a specific inductance of $\ell = 600$fH$/\mu$m [53]. The junction capacitances and resistances were as follows: (filled triangles) $C_0 = 5$fF, $R_0 = 29$kΩ, (filled circles) $C_0 = 6.5$fF, $R_0 = 8.8$kΩ, (open triangles) $C_0 = 4$fF, $R_0 = 23$kΩ, (open squares) $C_0 = 3$fF, $R_0 = 27$kΩ, (open circles) $C_0 = 3$fF, $R_0 = 11$kΩ.

To improve the agreement between the model and the data, we have artificially increased the temperature from the experimental value of 4.2K to 20K.

These assumptions have some strengths and weaknesses. The main strength is having the impedance of the leads be close to the impedance of free space rather than the resistance quantum. In this case we do not need to invoke a resistance that is not measured. This low impedance also gives zero-bias anomalies of about the right size. In the present model at least, much larger anomalies would be expected for the larger impedance environment. The energy scales in the model are the temperature and the energy at which the low-temperature power-law behavior crosses over to a Lorentzian behavior. For low impedance leads this is *much* larger than the charging energy. For charging energies between 1 and 50 meV, the temperature seems to be the only important energy scale, but for lower charging energies 0.1 to 0.5 meV both the temperature and the "Lorentzian" scales are important. By decreasing the charging energy it is possible to make the blockade go away as it does in the experiment as the wires get pushed closer together.

One obvious weakness is the large temperature required to produce the observed widths of the zero-bias anomalies. We do not believe that heating is important in the measurement, but if the present model approximates the important physics of the measurement there must be something we have not included that would broaden the spectra. One possible broadening mechanism would be the finite junction resistance that we have otherwise ignored. In the spin-wave model the finite junction resistance would be irrelevant as it is much larger than the impedance of the leads. In an improved model which correctly describes the discrete nature of the quantum charge fluctuations between the wires, a transition to Fermi liquid-like behavior might well occur as the junction resistance approaches R_H from above.

Another weakness is that the range of capacitances is larger than would be expected for two crossed cylinders that are much closer than their radii. One possible explanation for this range is that the wires are close enough that the roughness of the surfaces might strongly affect the junction properties. Another possible explanation is that the finite junction resistance effects that we have not included decrease the size of the blockade as they become

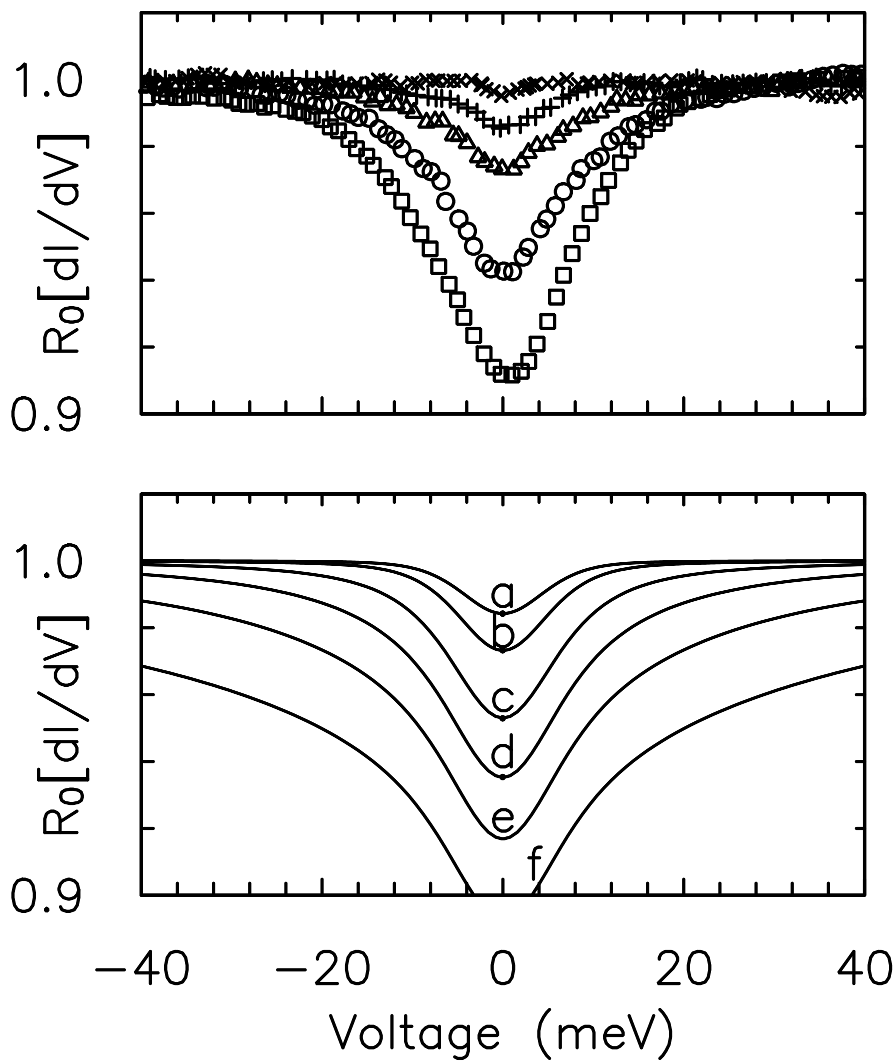

FIG. 8. Comparison of theory and the measurements of Gregory [22]. The bottom panel shows the calculated conductivity of isolated junctions with a series of different charging energies (E_C= (a) 0.1, (b) 0.2, (c) 0.5, (d) 1.0, (e) 2.0, (f) 5.0 meV) plotted as a function of voltage. The details of the calculation are discussed in the text. The top panel shows the measured conductivity for five different junctions.

more important.

Finally the long tails that are seen in the results of the model calculation are not seen in the experimental data. We suspect that the tails that would be present in the data are removed by the subtraction of the quadratic background.

VII. MICROWAVE-DRIVEN SINGLE JUNCTIONS

In the system of tunnel junction plus leads with a voltage bias the tunneling events are at most only weakly correlated in time. In a current biased system, on the other hand, the tunneling events are strongly correlated to each other, leading to the so-called single-electron tunneling (SET) oscillations [1]. To detect these oscillations, microwaves can be applied to the junction. The microwave frequency phase locks to the SET oscillations and produces constant-current steps in the I-V curve [13,14]. It is hoped that this effect in the future may provide a means for maintaining a standard of the Ampère, in much the same way as the constant-voltage steps of irradiated Josephson-junctions today are used to maintain a standard for the Volt. Calculations of these effects are based on semiclassical models of the Coulomb behavior which assume that the blockade energy is always the classical charging energy, $e^2/2C_0$. However, as we have seen, a quantum mechanical calculation for an isolated tunnel junction shows that the blockade energy is not constant but is distributed over a continuum of values that depends on the impedance of the leads. These quantum mechanical calculations [19–21] all treated voltage biased systems. In reality, the impedance of the leads connecting the tunnel junction to the voltage or current source implies that the junction "sees" a mixture between the two kinds of sources as illustrated by the following simple example.

A. Current– or Voltage Bias?

Usually the circuit containing the tunnel junction is assumed to be biased by a constant voltage– or current source. For a discussion of charging effects, however, the more relevant quantity is the actual voltage or current at the junction itself. In general there is a difference, which is usually neglected (as it is in this paper) for convenience rather than for being unimportant. In this section however, we shall illustrate the difference between ideal and actual bias by considering two simple examples. In the first we consider a "current source" which in practice is a *voltage* source V_s with a large but finite internal resistance R_s. As a function of the bias current we demonstrate that the tunnel junction "sees" a crossover from a current bias regime at high currents to a voltage bias regime at low currents. The crossover current is inversly proportional to R_s, and is zero only in the limit of infinitely large internal resistance of the source. In the second example we use a simple circuit to demonstrate that even if the circuit containing the tunnel junction could be attached to a *perfect* current source, the current through the junction would *not* be perfectly constant. The reason is a transient behaviour after each tunneling event caused by impedance mismatch between the various parts of the circuit.

Consider now the first example of a current source realized as a *voltage* source V_s in series with a large resistance R_s. Let us add the transmission line impedance which we simply assume to be a resistor R_L. The voltage source in series with this resistance is now attached to the junction (a capacitance C_0 and resistance R_0 in parallel). Using the classical equations of motion the time-dependent voltage across the junction in between tunneling events is

$$V_s = (R_s + R_L)\dot{q}_0 + q_0/C_0. \tag{55}$$

Solving for the voltage $V_0 = q_0/C_0$ across the junction, one finds

$$V_0(t) = V_s + [V_0(0) - V_s]\exp(-t/(R_s + R_L)C_0). \tag{56}$$

Suppose now that we have a "perfect" current source: $R_s \to \infty$ and $V_s \to \infty$ while $V_s/R_s = I_s$ is constant. Then we have (returning to the charge variable)

$$q_0(t) = q_0(0) + I_s t, \tag{57}$$

which is what we expect for the current bias. Suppose on the other hand that V_s and $R_s \gg R_L$ are finite. Then, if V_s is just above the threshold value for tunneling, V_0 will reach this value roughly a (finite) time $R_s C_0$ after a tunneling event. But now, in the semiclassical picture, the probability for tunneling will be small and the voltage across the junction will remain at V_s for a long time. Hence we in effect are in a voltage biased situation. For larger values of V_s, tunneling probability is larger and tunneling will occur on the average after a time shorter than $R_s C_0$. We can then expand the exponential function and get a junction charge which grows linearly with time in between tunneling events. Gradually we will cross over into a current biased regime. The crossover will occur for

$$e/I \approx R_s C_0, \tag{58}$$

for the circuit we are discussing now. For sufficiently small currents one would then always be in a voltage biased regime.

In our second example we use a slightly more complicated circuit. Let the junction as before be represented by C_0 and R_0 in parallel, but instead of a single resistor the leads are now represented by an R_L, C_L element. The source is still a voltage source V_s in series with a large resistance R_s. The solution for the time development of the junction voltage in between tunnelings is straightforward. The full solution is somewhat complicated. If for simplicity we let $\dot{V}_0(0) = 0$, one finds

$$\begin{aligned} V_0(t) = V_s + \frac{V_s - V_0(0)}{\alpha_+ - \alpha_-}\{\alpha_- \exp(-\alpha_+ t) - \\ \alpha_+ \exp(-\alpha_- t)\} \end{aligned} \tag{59}$$

where

$$\begin{aligned} \alpha_\pm &= a \pm \sqrt{a^2 - b} \\ a &= (1/\tau_{L0} + 1/\tau_{LL} + 1/\tau_{sL})/2 \\ b &= (1/\tau_{sL})(1/\tau_{L0}) \end{aligned} \tag{60}$$

and

$$\tau_{L0} = R_L C_0, \quad etc. \tag{61}$$

If we now take the "current bias" limit V_s, $R_s \to \infty$ while $V_s/R_s = I_s$ the result simplifies. Assuming $C_L \gg C_0$ one has (switching to the charge variable)

$$q_0(t) = q_0(0) + \frac{C_0}{C} I\{t - \tau\,(1 - \exp(-t/\tau))\}, \tag{62}$$

where $\tau = R_L C_0$. Hence after a tunneling there is some transient behavior on the time scale $R_L C_0$ after which we are back into a current biased situation. Even with a perfect current source, the current through the tunnel junction is not constant.

Below we neglect these complications and present a quantum mechanical calculation of a voltage bias system with applied microwaves. At the same time we describe how the results we discuss in Sec. 4 can be derived in perturbation theory [20]. Ultimately one would like to treat a current biased junction with applied microwaves.

B. Microwave-Driven, Voltage-Biased Junction

The non-interacting part of the Hamiltonian is

$$H_0 = \sum_m \epsilon_m c_m^\dagger c_m + \sum_n \epsilon_n d_n^\dagger d_n + \sum_k \hbar\omega_\alpha a_\alpha^\dagger a_\alpha, \tag{63}$$

where the operators, c_m, and d_n describe the left and right electron seas (leads) respectively, and the electromagnetic modes of the environment are described by a_α, the creation and destruction operators for the normal modes of the electromagnetic Hamiltonian. The tunneling of electrons from side to side is governed by the tunneling Hamiltonian

$$H_T = T_{RL} + T_{LR}, \tag{64}$$

where

$$T_{RL} = {T_{LR}}^\dagger = \sum_{m,n} T_{m,n} c_m d_n^\dagger \mathrm{e}^{iep_0/\hbar}. \tag{65}$$

The electron operators transfer an electron from one side to the other, ignoring the charge of the electron. The charge of the tunneling electron is accounted for by the final exponential in

Eq.(65), which is the displacement operator discussed in connection with the single-oscillator pedagogical model. It changes the charge on the junction capacitor by e.

For junctions not driven by microwaves, the initial state is the ground state of the system, $|0\rangle$, which consists of Fermi seas in the left and right electron baths, and no excited bosons. In a finite temperature calculation, a set of states taken from a thermal ensemble would be averaged over. Operating on the initial state with the tunneling part of the Hamiltonian transfers an electron from below the Fermi sea on one side to above the Fermi sea on the other and displaces the electromagnetic modes. The final states are thus a hole in one Fermi sea, an electron above the other Fermi sea, and the displaced electromagnetic modes. Energy conservation says that the energy lost by the electron (due to the applied voltage the hole sits at a higher energy than the electron), is absorbed by the boson modes.

While applying microwaves to a junction complicates the physics considerably, the main effect of the microwaves is to induce an oscillating voltage and hence charge across the junction. The simplest way to model the oscillating charge in the transmission line model is to assume that all of the transmission line modes are in their ground state except one. The mode with the same frequency as the microwave field is macroscopically occupied in a coherent state. This state is generated by the translation operator for the macroscopically occupied mode, denoted by d, applied to the ground state of the transmission line (we let $\hbar = 1$ below),

$$|AC\rangle = \mathrm{e}^{ieAp_d}|0\rangle. \tag{66}$$

Under the influence of the non-interacting Hamiltonian, the charge and hence the voltage on the junction oscillates at the driving frequency of the microwaves, ω_d. The AC voltage across the junction is related to the amplitude of the displaced charge,

$$\begin{aligned} \langle\, q_0(t)\rangle \; &= \langle AC|\mathrm{e}^{iH_0t}q_0\mathrm{e}^{-iH_0t}|AC\rangle \\ &= -eA(\delta p_d)\cos\omega_d t \\ &= -C_0V_{AC}\cos\omega_d t, \end{aligned} \tag{67}$$

where δp_d is the expansion of p_0 in normal modes (*cf* Eq.(21)),

$$\delta p_d = \frac{C_0}{c}\sqrt{\frac{4}{1+(\omega_d\tau_d)^2}} \tag{68}$$

for an ideal transmission line. Note that ω_d is the driving frequency and τ_d is the discharge time. If the amplitude of the microwave field is set to zero, this initial state reduces to the initial state for the undriven junction.

We consider the effect of tunneling in time-dependent first-order perturbation theory, complicated by the fact that the initial state is not an eigenstate of the non-interacting Hamiltonian. Because we are interested in a state that has the form Eq.(66) at $t = 0$, we have to choose the initial state carefully. Since the initial state (microwave field) oscillates with a period, $T = 2\pi/\omega_d$, we take the limit that the initial time goes to negative infinity by making the initial time $-NT$, and take the limit that integer N goes to infinity. Using this procedure to maintain the phase at $t = 0$ gives for the state at time t,

$$\begin{aligned} |t\rangle &= \mathrm{e}^{-iH(t+NT)}|AC\rangle \\ &\approx \mathrm{e}^{-iH_0 t}\left[1 - i\int_{-NT}^{t} dt' H_T(t')\mathrm{e}^{\eta(t'-t)}\right] \\ &\quad \mathrm{e}^{-iH_0 NT}|AC\rangle. \end{aligned} \tag{69}$$

where the factor $\mathrm{e}^{\eta(t'-t)}$ insures that the interaction is turned on adiabatically in the distant past. Because N is an integer, when the last exponential factor operates on the initial state, $|AC\rangle$, it just returns the initial state. Taking the limit that $N \to \infty$ then gives,

$$\lim_{N\to\infty} |t\rangle \approx \mathrm{e}^{-iH_0 t}\left[1 - i\int_{-\infty}^{t} dt' H_T(t')\mathrm{e}^{\eta(t'-t)}\right]|AC\rangle. \tag{70}$$

The time dependence of the tunneling Hamiltonian, H_T, is that due to the non-interacting Hamiltonian.

The perturbed state is used to calculate the expectation value of the current operator

$$I = ie(T_{RL} - T_{LR}), \tag{71}$$

which is related to the tunneling Hamiltonian. Keeping only those terms with two factors of the interaction Hamiltonian yields

$$\langle\, I(t)\rangle \;= \\ 2e\mathrm{Re}\left[\int\limits_{-\infty}^{t} dt' \mathrm{e}^{\eta(t'-t)}\langle AC|[T_{RL}(t'),T_{LR}(t)]|AC\rangle\right] \tag{72}$$

The time dependence of the current operator comes from the left-most exponential in Eq.(70). Substituting the forms of the tunneling Hamiltonian into the expression for the current and taking the electron expectation values yields

$$\langle\, I(t)\rangle \;= 2e\int\limits_{-\infty}^{t} dt' \sum_{m,n}\mathrm{Re}\Big[\mathrm{e}^{i(\epsilon_n-\epsilon_m-i\eta)(t'-t)}|T_{m,n}|^2 \\ \Big\{[1-n_l(\epsilon_m)]n_r(\epsilon_n)\langle AC|\mathrm{e}^{iep_0(t')}\mathrm{e}^{-iep_0(t)}|AC\rangle - \\ n_l(\epsilon_m)[1-n_r(\epsilon_n)]\langle AC|\mathrm{e}^{iep_0(t)}\mathrm{e}^{-iep_0(t')}|AC\rangle\Big\}\Big]. \tag{73}$$

The functions n_r and n_l are Fermi functions for the left and right electrons seas; one or the other could be at a raised voltage. We are assuming that (quasi-) equilibrium is being maintained in both electron seas by some scattering process that has not been explicitly included in the Hamiltonian. This is consistent with treating the tunneling in first order perturbation theory assuming that the tunneling matrix elements are small. If the tunneling resistance is much greater than the resistance quantum, $R_H = h/e^2$, it is generally believed that the electron seas behave as if they were decoupled and equilibrate separately from each other.

The expectation values of the electromagnetic bosons in the expression for the current can be computed using the following expression

$$\langle 0|\mathrm{e}^{-i\mathrm{A}}\mathrm{e}^{i\mathrm{B}}\mathrm{e}^{-i\mathrm{C}}\mathrm{e}^{i\mathrm{A}}|0\rangle = \langle 0|\mathrm{e}^{i\mathrm{B}}\mathrm{e}^{-i\mathrm{C}}|0\rangle\mathrm{e}^{[\mathrm{A},\mathrm{B}]}\mathrm{e}^{[\mathrm{C},\mathrm{A}]}, \tag{74}$$

which is valid if the commutators contained in it are c-numbers. Using this expression the expectation value splits into two factors:

$$\langle AC|\mathrm{e}^{\pm iep_0(t)}\mathrm{e}^{\mp iep_0(s)}|AC\rangle = \langle 0|\mathrm{e}^{\pm iep_0(t)}\mathrm{e}^{\mp iep_0(s)}|0\rangle$$

$$\mathrm{e}^{\pm e^2 A(\delta p_d)[p_d, p_d(t)]}$$

$$\mathrm{e}^{\mp e^2 A(\delta p_d)[p_d, p_d(s)]}. \tag{75}$$

The first factor, which is still an expectation value, is independent of the time dependent voltage. It is related to the shake-up excitation spectrum, $A(\omega)$ as is discussed below. The next two factors are due to the stimulated emission and absorption of the microwaves.

As discussed above in Eq.(27), the first factor represents the singular excitation of the electromagnetic modes by the tunneling event and can be written in terms of the the excitation spectrum, $A(\omega)$;

$$\langle 0|\mathrm{e}^{\pm iep_0(t)}\mathrm{e}^{\mp iep_0(s)}|0\rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\mathrm{e}^{-i\omega(t-s)}A(\omega). \tag{76}$$

This expression can be evaluated by making a cumulant expansion, which is exact for harmonic oscillators, and relating the resulting correlation function, $\langle 0|p_0(t)p_0(s)|0\rangle$, to the classical time dependence of the charge on the junction capacitor [21].

The second two factors in Eq.(75) are determined primarily by the microwave driving voltage and can be written in terms of a sum of Bessel functions

$$\mathrm{e}^{\pm e^2 A(\delta p_d)[p_d, p_d(t)]} = \sum_{\ell=-\infty}^{\infty} \mathrm{e}^{\mp i\ell\omega_d t} J_\ell(\alpha), \tag{77}$$

where $\alpha = eV_{AC}/\hbar\omega_d$, is the dimensionless driving voltage. We have made use of the relationship between the applied voltage and the excitation of the driven mode from Eq.(67).

Substituting the above expressions into Eq.(73) and doing one of the time integrals gives the following expression for the expectation value of the current as a function of time

$$\begin{aligned}\langle I(t)\rangle = e\sum_{m,n}|T_{m,n}|^2 \sum_{\ell=-\infty}^{\infty} J_\ell(\alpha) \sum_{k=-\infty}^{\infty} J_k(\alpha) \\ \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}A(\omega)\mathrm{Re}\Bigg[\frac{1}{i(\epsilon_n - \epsilon_m - \omega - \ell\omega_d - i\eta)} \\ \Big\{[1-n_l(\epsilon_m)]n_r(\epsilon_n)\mathrm{e}^{i(k-\ell)\omega_d t} - \\ n_l(\epsilon_m)[1-n_r(\epsilon_n)]\mathrm{e}^{-i(k-\ell)\omega_d t}\Big\}\Bigg].\end{aligned} \tag{78}$$

Note that the first and second terms give currents going from left to right and right to left respectively. Averaging the current over a period of the driving frequency gives the simple result,

$$\overline{\langle I \rangle} = \int_{t_0}^{t_0+T} \frac{dt}{T} \langle I(t) \rangle = \sum_{\ell=-\infty}^{\infty} J_\ell(\alpha)^2 I_0(eV - \ell\omega_d), \tag{79}$$

where I_0 is the current as a function of voltage with no driving voltage present given by Eq.(31). The microwaves modify the current in a way that it only depends on the microwave amplitude and the current in the absence of the microwaves as in Eq.(79).

This expression for tunneling currents in the presence of microwaves is quite general. It has been derived by both (among others) Tien and Gordon for a general junction [51] and Odintsov [52] for a resistively shunted junction exhibiting a Coulomb blockade. While those derivations treat the applied field classically and we treat the field quantum mechanically, the derivations are not that different. Those derivations that treat the fields classically find the discrete nature of the field-induced transistions because they calculate the time averaged current from a quantum mechanical treatment of the electrons. In our derivation, the quantum mechanical field behaves quite classically. This is because the field consists of a displaced ground state wavepacket whose centroid follows the classical equations of motion. The shape of the wave packet only changes as it would in the absence of the microwaves; the effect of the microwaves derives from the classical motion of the centroid. In Eq. (79), the quantum mechanical effects that give the blockade, contained in I_0, separate from the classical-like effects due to the microwaves, the Bessel function factors. This separation derives from the separation into two types of factors of Eq.(75). Neither of these derivations treat the back action of the tunneling event on the microwave field. In our derivation, such effects would show up in higher order perturbation theory; these contributions would be non-classical.

Note that if the microwave amplitude is set to zero, we recover the results for the undriven junction, because, $J_\ell(0)^2 = \delta_{\ell,0}$. If the tunneling matrix elements and the density of states are taken to be constant, the "bare" resistance of the junction is related to the transition matrix elements, $R_0^{-1} = 2\pi |T|^2 \rho^2 e^2/\hbar$, where ρ is the—constant— electron density of states in the leads. The inverse of this resistance appears as a prefactor in our result for the current, Eq.(31).

While the result, Eq. (79), can be derived from a field that behaves classically, it is still a quantum mechanical result, only discrete transitions are allowed. As the ratio of the AC voltage to the driving frequency, α, becomes large, we reach a classical limit in which the discreteness of the transitions becomes unimportant. In the classical limit the instantaneous current is expected to depend only on the instantaneous voltage. The average classical current is given by

$$\begin{aligned}\overline{I_{cl}} &= \int_{t_0}^{t_0+T} \frac{dt}{T} I\left(V + V_{AC}\cos(w_d t)\right) \\ &= [\int_{-1}^{1} du][\frac{1}{\pi}\frac{1}{\sqrt{1-u^2}}][I(V + V_{AC}u)]. \end{aligned} \tag{80}$$

The change of variables in the right hand part of this expression is made to suggest a parallel between it and a rewriting of the quantum result, Eq.(79),

$$\overline{I_{qu}} = [\sum_{\ell=-\infty}^{\infty} \frac{1}{\alpha}][\alpha J_\ell(\alpha)^2][I_0(eV + V_{AC}\frac{\ell}{\alpha})]. \tag{81}$$

Given this parallel structure the classical limit is obtained from the asymptotic behavior of Bessel functions,

$$\begin{aligned}\alpha J_\ell(\alpha)^2 \approx & \frac{1}{\pi}\frac{1}{\sqrt{1-(\frac{n}{\alpha})^2}} \\ & \left[\, 2\cos^2\left(\alpha\sqrt{1-(\frac{n}{\alpha})^2} - n\sec^{-1}(\frac{n}{\alpha}) - \frac{\pi}{4}\right)\right], \end{aligned} \tag{82}$$

for $n \ll \alpha$. The limit that $eV_{AC}/\hbar\omega_d = \alpha \to \infty$ corresponds to an infinite number of electromagnetic bosons being absorbed and through the correspondence principle is the classical limit. In this limit, the quantity in square braces in Eq.(82) oscillates more and more rapidly and can be averaged over some fine energy scale to give the classical limit.

The behavior of these systems, at zero temperature, is determined by the ratio of the impedance of the leads to the quantum resistance, $g = 2Z/R_H$ ($R_H = h/e^2$), and ratios of the three independent energies in the problem, the charging energy of the junction, $E_C = e^2/2C_0$, the energy corresponding to the applied microwave voltage, eV_{AC}, and the energy of the microwave photons, $\hbar\omega_d$. For high impedance leads, the blockade (in the absence of microwaves) behaves like a slightly smeared semiclassical blockade, as seen in the top panel of Fig. 9. The behavior of this junction in the presence of microwaves is shown in Fig. 9 in terms of the conductivity at zero DC bias as a function of the AC bias, V_{AC}. Scaling the AC bias by the microwave frequency, $\alpha = eV_{AC}/\hbar\omega_d$, highlights the limiting behavior for large values of the photon energy compared to the charging energy, $\hbar\omega_d/E_C$. On the other hand, scaling the AC bias by the charging energy, eV_{AC}/E_C, highlights the limiting semi-classical behavior for large values of α.

As can be seen from Eq.(79), the conductivity is determined by sampling the DC conductivity at a series of voltages separated by the microwave energy. If the microwave energy is much larger than the charging energy then the conductivity is zero for the zero photon term and one for all others. Thus, in the limit that $\hbar\omega_d/E_C$, becomes large, the DC conductivity plotted as a function of $\alpha = eV_{AC}/\hbar\omega_d$ is just $1 - J_0(\alpha)^2$, independent of $\hbar\omega_d/E_C$. This limiting behavior holds for all values of α.

The semi-classical result is just the fraction of each period over which the classical voltage has a greater magnitude than the charging energy, $(2/\pi)\cos^{-1}(E_C/eV_{AC})$ for $eV_{AC}/E_C > 1$. This limit is approached when the applied voltage is much larger than the driving frequency, $eV_{AC}/\hbar\omega_d \gg 1$. When eV_{AC} is scaled by E_C, the conductivity approaches the semi-classical result over a larger range of voltages the smaller $\hbar\omega_d/E_C$ is. For the systems considered here, these curves will not converge to the semiclassical result because the I-V curve in the absence of microwaves differs from the semi-classical result. For a given junction and a given microwave field, $eV_{AC}/E_C < 1$, the largest response to the microwaves comes about when the microwave frequency is equal to the charging energy, $\hbar\omega_d/E_C = 1$. For higher frequencies, V_{AC} is not large enough to excite many photons, and the response falls off. For lower frequencies, the classical limit is approached where the response is zero for $eV_{AC}/E_C < 1$.

For lower impedance leads, the behavior is more complicated because the charging energy

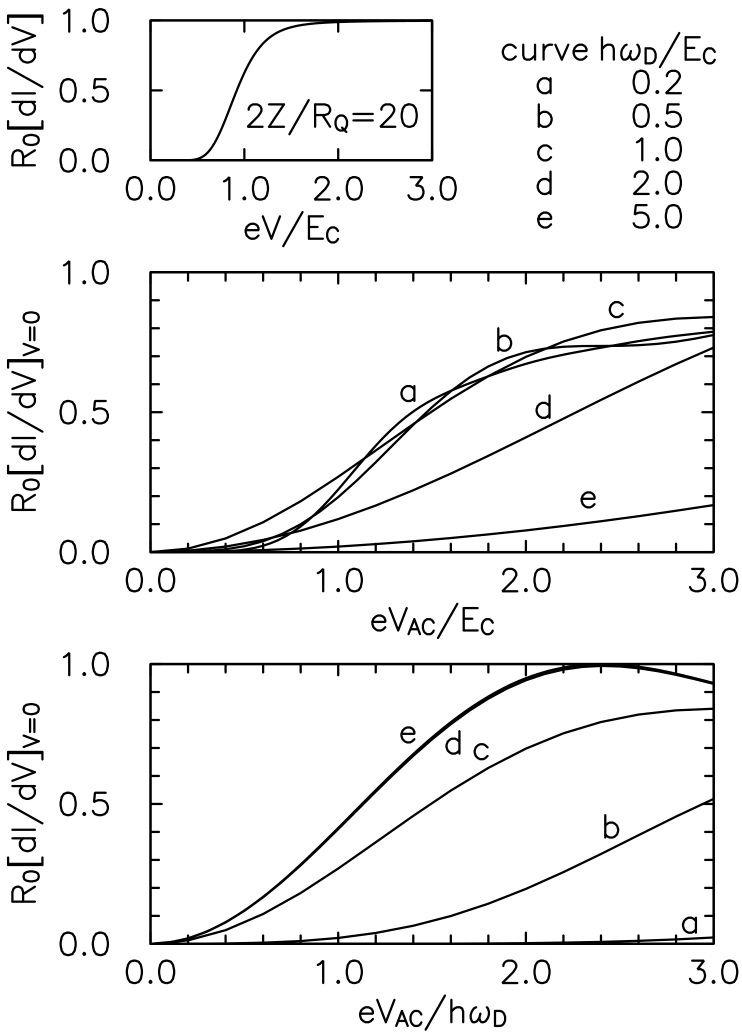

FIG. 9. Microwave driven junctions. The top panel shows the conductivity as a function of DC bias for the junction considered here. The other two panels show the DC conductivity at zero DC bias as a function of AC bias for a series of different AC frequencies. The same curves are plotted in both figures, the AC bias is scaled by different values in each. The ratio of the AC frequency to the charging energy of the junction is given in the upper right hand part of the figure. The lower panel shows that as the AC frequency becomes much larger than the charging energy that the conductivity approaches a limiting form $(1 - J_0(\alpha)^2)$ as a function of the AC voltage scaled by the AC frequency. The middle panel shows that in the opposite limit that the conductivity approaches the classical limit.

is no longer the appropriate energy scale to describe the behavior of the conductivity as a function of applied DC voltage. AC voltages much higher than E_C/e are needed to reach the limit where the conductivity only depends on $\alpha = eV_{AC}/\hbar\omega_d$. Since the DC I-V curves are very different than the semi-classical result, the semi-classical limit is never approached, even for large values of $eV_{AC}/\hbar\omega_d$. While resistive leads have I-V spectra that looks close to semiclassical behavior, the long tails that are present complicate any simple limits.

VIII. SUMMARY

We have discussed the effect of the electromagnetic environment on tunneling in isolated tunnel junctions. We find that there is a big difference between what would be expected from a semiclassical analysis and what we find from our quantum mechanical analysis. In particular, if the leads attached to the junction have low impedances, then the Coulomb blockade of the tunneling is smeared out. These expectations are born out by comparison with three recent experiments.

By considering first a pedagogical model consisting of a single harmonic oscillator, and then a more detailed model of a transmission line, we have shown that a tunneling electron excites an infinite number of low energy electromagnetic bosons. The excitation of an infinite number of bosons leads to power-law behavior, $dI/dV \sim V^g$ for the differential conductivity, where the coupling constant, $g = 2Z/R_H$, the ratio of the impedance of the leads to the quantum resistance, $R_H = h/e^2$. Thus, the behavior of the tunnel junctions at small voltages depends crucially on the impedance of the environment.

We have extended the results to finite temperatures, and to more complicated leads, including resistive leads, and leads with discontinuities in their properties. We have discussed the important approximations made in this calculation and the possible consequences of relaxing them. These approximations are: 1) the degrees of freedom can be separated into microscopic single-particle modes and macroscopic collective modes, 2) the tunneling event is instantaneous on the important time scales, and 3) that the "bare" junction resistance is high enough that tunneling events are uncorrelated.

This model was used to analyze experiments on single aluminum/aluminum-oxide tunnel-junctions by Delsing *et al.* [16] and Cleland *et al.* [18], as well as on junctions formed by crossed platinum wires (Gregory [22]) separated by a frozen helium. The model qualitatively accounted for the zero-bias anomalies found in these measurements, particularly the size of the anomaly. We suggested reasons why the agreement was not perfect in all cases. The agreement between theory and experiment demonstrates that this model contains the essential physics necessary to explain the effect of the environment on tunneling.

Finally we discussed the difference between current–and voltage bias and considered the effects of microwave radiation on voltage biased junctions. We concluded that even if the circuit is configured to be current biased, that for low enough currents, the junction behaves as if it were voltage biased, the regime in which these calculations are valid. When microwaves are applied to the junctions we found the regimes in which a classical treatment of the microwaves would be valid, depending on the coupling constant, $g = 2Z/R_H$, and the three independent energy scales in the problem.

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