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Re-visiting the Balazs Thought Experiment in the Presence of Loss: Electromagnetic-Pulse-Induced Displacement of a Positive-Index Slab having Arbitrary Complex Permittivity and Permeability

Kenneth J. Chau ⋆**, Henri J. Lezec**

Center for Nanoscale Science and Technology, National Institute of Standards and Technology, Gaithersburg, Maryland, USA

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Abstract Over a half-century ago, Balazs proposed a thought experiment to deduce the form of electromagnetic momentum in a lossless and non-dispersive slab by imposing conservation of global momentum and system center-of-mass velocity after a pulse has traveled through the slab. Here, we re-visit the Balazs thought experiment by explicit calculations of momentum transfer and center-of-mass displacement of a non-dispersive, positive-index slab of arbitrary complex permittivity and permeability using a set of postulates consisting only of Maxwell's equations, a generalized Lorentz force law, the Abraham form of the electromagnetic momentum density, and conservation of both pulse and slab mass. In the case where the slab is lossless, we show that a pulse of arbitrary shape incident onto the slab conserves both global momentum and system center-of-mass velocity, consistent with the starting postulates of the Balazs thought experiment. In the case where the slab is lossy, we show, within the context of the above postulates, that global momentum is always conserved and that system center-of-mass velocity is conserved only when mass transfer from the pulse to the slab is described by an incremental pulse-mass-transfer model, proposed here, in which the pulse deposits mass in the slab with a distribution corresponding to the instantaneous mass density profile of the pulse.

⋆ *Present address:* School of Engineering, University of British Columbia, Kelowna, British Columbia, Canada

1 Introduction

Light is known to carry momentum [1] and apply forces when interacting with matter [2,3]. It is well-established that the momentum of a pulse of light of energy E in free space is $p_0 = E/c_0$, where c_0 is the speed of light. The momentum of a pulse of light of the same energy E propagating in a ponderable medium is not trivially derived and remains a topic of debate. Consider, for example, the case of a non-dispersive, lossless medium of positive refractive index n. Compelling arguments exist either in favor of a Minkowski form [4] of the pulse momentum, given by np_0, or an Abraham form [5] of the pulse momentum, given by p_0/n.

The elegant Balazs thought experiment [6] has allowed inference of the value of the momentum of a pulse of light in a lossless, non-dispersive dielectric medium. The Balazs thought experiment consists of two identical enclosures as shown in Fig. 1, each composed of a region of vacuum encompassing two entities: a non-dispersive, lossless, and initially-stationary dielectric slab having refractive index n, thickness L, and mass M and a finite-duration electromagnetic pulse with free-space mass $m = E/c_0^2$ initially traveling in vacuum at c_0. In enclosure 1, the pulse propagates along a straight path through only vacuum, and in enclosure 2, the pulse propagates along a straight path through both vacuum and the slab. It is assumed that the slab is massive and impedance-matched to free-space and that the pulse is a plane-wave normally incident onto the slab, traveling with velocity c_0 in vacuum and c_0/n in the slab. In the absence of external forces, the total momenta in enclosures 1 and 2 are conserved and the centers-of-mass of the two systems move with identical, uniform velocities. The global momentum in enclosure 1 for all time is the momentum of the pulse in vacuum, mc_0, and the center-of-mass of the system in enclosure 1 moves with velocity $mc_0/(m + M)$. In enclosure 2, analysis of the slab center-of-mass displacement before and after the pulse has interacted with the slab, along with the requirement of conservation of system center-of-mass velocity and invariance of the pulse mass, implies that the slab acquires a momentum $(1 - 1/n)mc_0$ while the pulse is fully contained in the slab. For global momentum to be conserved, the momentum of the pulse in the slab must then be mc_0/n, corresponding to an Abraham form of the pulse momentum [5] (as opposed to, notably, a Minkowski form expressed as nmc_0 [4]). A summary of the assumptions and postulates of the Balazs thought experiment are listed in Table 1. It should be noted that the Balazs thought experiment is a before/after experiment which requires full emergence of the pulse from the slab and does not explicitly consider dynamics while the pulse is in the slab, which would be necessary to deal with the case of a partially or completely absorbing slab.

Here, we re-visit the Balazs thought experiment starting with four postulates: the microscopic Maxwell's equations, a generalized Lorentz force law applicable for media with both electric and magnetic responses [8–13], the Abraham form of the electromagnetic momentum density, and conservation of both slab and pulse mass, where the latter is assumed to be $m = E/c_0^2$ in both vacuum and the slab. We consider an electromagnetic pulse traveling through a massive, non-dispersive, and positive-index slab of arbitrary relative permittivity and permeability. For all times during the interaction of the pulse and the slab, we calculate the momenta and center-of-mass displacements of both the pulse and the slab. Two cases are considered: one in which the slab is lossless, matching the condition originally studied by Balazs and another in which the slab is lossy, a condition not considered by Balazs. For the case of a lossless slab, we analytically derive the slab center-of-mass and system center-of-mass displacements due to a pulse of arbitrary shape and show that global momentum and system center-of-mass velocity are conserved, consistent with two of Balazs' starting postulates. For the case of a lossy slab, we numerically model the slab and system displacement for varying degrees of loss up to the limiting case where the incident pulse is completely absorbed by the slab. Absorption in the slab is described by an incremental pulse-mass transfer model, proposed here, in which the pulse deposits mass in the slab according to the instantaneous mass density profile of the pulse. It is shown that global momentum is conserved for all time, regardless of the degree of loss in the slab, and that the system center-of-mass velocity is conserved only when pulse mass deposition in the slab is described by the proposed incremental pulse-mass-transfer model.

2 Analytical Calculations of an Electromagnetic Pulse Incident onto a Lossless Slab

We analytically treat the case of a pulse with arbitrary temporal profile incident onto a lossless slab to directly compare with the results of the Balazs thought experiment and to establish basic concepts to be used in later sections. Similar to the Balazs thought experiment, we consider a non-dispersive, lossless material. We treat the more explicit case of a right-handed material with arbitrary electric and magnetic responses, modeled by real, relative permittivity $\epsilon_r > 0$ and real, relative permeability $\mu_r > 0$, yielding a positive refractive index $n = \sqrt{\epsilon_r \mu_r} > 0$. Like the Balazs thought experiment, we assume that the slab is impedance-matched to vacuum, which in our case can be achieved by setting $\epsilon_r = \mu_r$ to yield a relative impedance $\eta_r = \sqrt{\mu_r/\epsilon_r} = 1$.

The system consists of a region of vacuum encompassing an initially-motionless, massive slab and an electromagnetic pulse. We restrict our analysis to the non-relativistic limit by choice of a slab mass $M >> m$ such that the center-of-mass velocity of the slab is significantly less than c_0. The vacuum region is characterized by a free-space permittivity $\epsilon = \epsilon_0$ and a free-space permeability $\mu = \mu_0$. The slab is composed of a non-dispersive, lossless medium characterized by a permittivity $\epsilon = \epsilon_r \epsilon_0$ and a permeability $\mu = \mu_r \mu_0$.

First, we postulate that the propagation of the pulse is governed by the microscopic Maxwell's equations. The electric field $\mathbf{E}$ and the magnetic field

$\mathbf{H}$ of the pulse are then related (in time-domain notation) by Ampere's Law

$$\nabla \times \mathbf{H} = \frac{\partial \epsilon \mathbf{E}}{\partial t}, \tag{1}$$

and by Faraday's Law

$$\nabla \times \mathbf{E} = -\frac{\partial \mu \mathbf{H}}{\partial t}. \tag{2}$$

The linear medium response to the electric and magnetic fields of the pulse yields an electric displacement field

$$\mathbf{D} = \epsilon \mathbf{E}, \tag{3}$$

a magnetic flux density field

$$\mathbf{B} = \mu \mathbf{H}, \tag{4}$$

an electric polarization field

$$\mathbf{P} = (\epsilon - \epsilon_0)\mathbf{E}, \tag{5}$$

and a magnetic polarization (magnetization) field

$$\mathbf{M} = (\mu - \mu_0)\mathbf{H}. \tag{6}$$

Temporal variation in $\mathbf{P}$ and $\mathbf{M}$ yield an electric current density $\mathbf{J}_e$ and a magnetic current density $\mathbf{J}_m$ given respectively by

$$\mathbf{J}_e = \frac{\partial \mathbf{P}}{\partial t} \tag{7}$$

$$= \frac{\partial \mathbf{D}}{\partial t} - \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{8}$$

and

$$\mathbf{J}_m = \frac{\partial \mathbf{M}}{\partial t} \tag{9}$$

$$= \frac{\partial \mathbf{B}}{\partial t} - \mu_0 \frac{\partial \mathbf{H}}{\partial t}. \tag{10}$$

The electromagnetic pulse propagates in the enclosure along the $+z$ direction and is directed at normal incidence onto the slab, which occupies the region $0 < z < L$. The electromagnetic pulse in the vacuum region $z < 0$ consists of an electric field

$$\mathbf{E}_0(z,t) = E_0 h(z - c_{p0}t) g(z - c_{g0}t)\,\hat{x}, \tag{11}$$

and a magnetic field

$$\mathbf{H}_0(z,t) = \frac{E_0}{\eta_0} h(z - c_{p0}t) g(z - c_{g0}t)\,\hat{y}, \tag{12}$$

where E_0 is the electric field amplitude and $\eta_0 = \sqrt{\mu_0/\epsilon_0}$ is the impedance of vacuum. We employ general field descriptions where $h(z - c_{p0}t) \equiv h(z,t)$

is an arbitrary function describing the spatially and temporally harmonic carrier wave propagating in the $+z$-direction with phase velocity c_{p0}, and $g(z - c_{g0}t) \equiv g(z,t)$ is an arbitrary function describing the electromagnetic pulse envelope propagating in the $+z$-direction with group velocity c_{g0}. The functions h and g are restricted only in that they must be consistent with Maxwell's equations.

It is well-established and uncontroversial that the electromagnetic momentum density in vacuum is given by [1]

$$\mathbf{G}_0(z,t) = \frac{\mathbf{E}_0(z,t) \times \mathbf{H}_0(z,t)}{c_0^2}, \tag{13}$$

where $\mathbf{E}_0(z,t)$ and $\mathbf{H}_0(z,t)$ are the electric and magnetic fields in vacuum. For a pulse that is fully located in the vacuum region, the total pulse momentum-per-unit-area at a given time t is obtained by integrating 13 yielding

$$\mathbf{p}_{p,A}(t) = \int_{-\infty}^{\infty} \frac{\mathbf{E}_0(z,t) \times \mathbf{H}_0(z,t)}{c_0^2} dz. \tag{14}$$

We evaluate the total pulse momentum-per-unit-area for full passage of the pulse through a plane at a given position z in vacuum by changing the integration variable in 14 to yield

$$\mathbf{p}_{p,A}(z) = \int_{-\infty}^{\infty} \frac{\mathbf{E}_0(z,t) \times \mathbf{H}_0(z,t)}{c_0^2} (c_0 dt) = \frac{1}{c_0} \int_{-\infty}^{\infty} \mathbf{E}_0(z,t) \times \mathbf{H}_0(z,t) dt. \tag{15}$$

Substitution of 11 and 12 evaluated at $z = 0^-$ into 15 yields

$$\mathbf{p}_{p,A}(0^-) \equiv p_0\, \hat{z} = \frac{E_0^2}{c_0 \eta_0} \int_{-\infty}^{\infty} h^2(0^-,t) g^2(0^-,t) dt\, \hat{z}, \tag{16}$$

where p_0 is the magnitude of the pulse momentum-per-unit-area in vacuum.

Solving 1 and 2 and imposing continuity of the net electric and magnetic fields at the interface yields a transmitted electromagnetic pulse in the slab having electric and magnetic fields

$$\mathbf{E}_t(z,t) = \kappa E_0 h(z - c_p t) g(z - c_g t)\, \hat{x}, \tag{17}$$

and

$$\mathbf{H}_t(z,t) = \frac{\kappa E_0}{\eta_r \eta_0} h(z - c_p t) g(z - c_g t)\, \hat{y}, \tag{18}$$

respectively, where c_p is the magnitude of the phase velocity of the carrier wave in the slab, c_g is the magnitude of the group velocity of the pulse envelope in the slab, and κ is the transmission coefficient of the vacuum/dielectric interface given by

$$\kappa = \frac{2\eta_r}{\eta_r + 1}. \tag{19}$$

The response of the slab to the electric and magnetic fields of the pulse yields an electric displacement field

$$\mathbf{D}_t(z,t) = \epsilon_r\epsilon_0\kappa E_0 h(z-c_pt)g(z-c_gt)\,\hat{x}, \tag{20}$$

a magnetic flux density field

$$\mathbf{B}_t(z,t) = \frac{\mu_r\mu_0\kappa E_0}{\eta_r\eta_0} h(z-c_pt)g(z-c_gt)\,\hat{y}, \tag{21}$$

an electric polarization field

$$\mathbf{P}_t(z,t) = (\epsilon_r-1)\epsilon_0\kappa E_0 h(z-c_pt)g(z-c_gt)\,\hat{x}, \tag{22}$$

a magnetic polarization (magnetization) field

$$\mathbf{M}_t(z,t) = \frac{(\mu_r-1)\mu_0\kappa E_0}{\eta_r\eta_0} h(z-c_pt)g(z-c_gt)\,\hat{y}, \tag{23}$$

an electric current density

$$\mathbf{J}_e(z,t) = \frac{\partial}{\partial t}\left[(\epsilon_r-1)\epsilon_0\kappa E_0 h(z-c_pt)g(z-c_gt)\right]\hat{x}, \tag{24}$$

and a magnetic current density

$$\mathbf{J}_m(z,t) = \frac{\partial}{\partial t}\left[\frac{(\mu_r-1)\mu_0\kappa E_0}{\eta_r\eta_0} h(z-c_pt)g(z-c_gt)\right]\hat{y}. \tag{25}$$

We next consider the momentum-per-unit-area imparted by the pulse to the slab. We postulate that electromagnetic fields interact with ponderable media via a generalized Lorentz force law originally derived by Einstein and Laub [8], and then re-derived in great detail by Mansuripur [9]. The corresponding force density is given by

$$\mathbf{f} = (\mathbf{P}\cdot\nabla)\mathbf{E} + (\mathbf{M}\cdot\nabla)\mathbf{H} + \mathbf{J}_e\times(\mu_0\mathbf{H}) - \mathbf{J}_m\times(\epsilon_0\mathbf{E}). \tag{26}$$

When 26 is applied in our divergence-free configuration, it simplifies to

$$\begin{aligned}\mathbf{f}(z,t) &= \mathbf{J}_e(z,t)\times\mu_0\mathbf{H}(z,t) - \mathbf{J}_m(z,t)\times\epsilon_0\mathbf{E}(z,t) && (27)\\ &= \mathbf{f}_e(z,t) + \mathbf{f}_m(z,t) && (28)\end{aligned}$$

where $\mathbf{f}_e(z,t)$ is component of the force density associated with the electric current density $\mathbf{J}_e(z,t)$ and $\mathbf{f}_m(z,t)$ is the component of the force density associated with the magnetic current density $\mathbf{J}_m(z,t)$. In Appendix A, we will examine the implications of choosing another variation of a generalized Lorentz force law [11–13] for calculating the force density exerted by the electromagnetic fields.

The force density exerted by the pulse in the slab is calculated by substituting 17, 18, 24, and 25 into 28. Dropping the arguments in the functions h and g, we first develop the expression for $\mathbf{f}_e(z,t)$

$$\mathbf{f}_e(z,t) = \frac{(\epsilon_r-1)\kappa^2E_0^2}{c_0^2\eta_r\eta_0}\left(\frac{\partial h}{\partial t}g + \frac{\partial g}{\partial t}h\right)hg\,\hat{z}. \tag{29}$$

Using the relations

$$\frac{\partial h}{\partial t} = -c_p \frac{\partial h}{\partial z}, \tag{30}$$

and

$$\frac{\partial g}{\partial t} = -c_g \frac{\partial g}{\partial z}, \tag{31}$$

and because dielectric is non-dispersive such that $c_g = c_p = c_0/n$, where c_0 is the speed of light in free space, 29 can be expressed in terms of the spatial gradients of the fields

$$\mathbf{f}_e(z,t) = -\frac{(\epsilon_r - 1)\kappa^2 E_0^2}{nc_0\eta_r\eta_0} \left(\frac{\partial h}{\partial z} hg^2 + \frac{\partial g}{\partial z} h^2 g \right) \hat{z}. \tag{32}$$

Similar calculations for $\mathbf{f}_m(z,t)$ yield

$$\mathbf{f}_m(z,t) = -\frac{(\mu_r - 1)\kappa^2 E_0^2}{nc_0\eta_r\eta_0} \left(\frac{\partial h}{\partial z} hg^2 + \frac{\partial g}{\partial z} h^2 g \right) \hat{z}. \tag{33}$$

The total instantaneous force-per-unit-area $\mathbf{F}(t)$ exerted by the pulse onto the dielectric slab is determined by substituting 32 and 33 into 28 and integrating the resulting expression throughout the extent of the slab

$$\mathbf{F}(t) = -\frac{\kappa^2 E_0^2}{nc_0\eta_r\eta_0} (\epsilon_r + \mu_r - 2) \int_{0^+}^{L} \left(\frac{\partial h}{\partial z} hg^2 + \frac{\partial g}{\partial z} h^2 g \right) dz \; \hat{z}. \tag{34}$$

The integral expression in 34 can be developed as

$$\int_{0^+}^{L} \left(\frac{\partial h}{\partial z} hg^2 + \frac{\partial g}{\partial z} h^2 g \right) dz = \frac{1}{2} \left[-h^2(0^+,t) g^2(0^+,t) + h^2(L,t) g^2(L,t) \right], \tag{35}$$

such that 34 becomes

$$\begin{aligned} \mathbf{F}(t) = {} & \frac{\kappa^2 E_0^2}{2nc_0\eta_r\eta_0} (\epsilon_r + \mu_r - 2) \\ & \left[h^2(0^+,t) g^2(0^+,t) - h^2(L,t) g^2(L,t) \right] \hat{z}. \end{aligned} \tag{36}$$

Integrating 36 with respect to time yields the net momentum $\mathbf{p}_s(t)$ imparted by the pulse onto the dielectric slab at a given time t

$$\begin{aligned} \mathbf{p}_s(t) & \equiv p_s(t)\,\hat{z} \\ & = \frac{\kappa^2 E_0^2}{2nc_0\eta_r\eta_0} (\epsilon_r + \mu_r - 2) \\ & \quad \int_{-\infty}^{t} \left[h^2(0^+,\tau) g^2(0^+,\tau) - h^2(L,\tau) g^2(L,\tau) \right] d\tau \, \hat{z}, \end{aligned} \tag{37}$$

where $p_s(t)$ is the magnitude of the momentum-per-unit-area imparted to the slab.

We define a normalized momentum transfer, $\zeta(t) = p_s(t)/p_0$, as

$$\zeta(t) = \frac{\kappa^2}{2n\eta_r}(\epsilon_r + \mu_r - 2) \times \frac{\int_{-\infty}^{t}\left[h^2(0^+,\tau)g^2(0^+,\tau) - h^2(L,\tau)g^2(L,\tau)\right]\,d\tau}{\int_{-\infty}^{\infty} h^2(0^-,t)g^2(0^-,t)dt}. \tag{38}$$

Simplification of the pre-factor in 38 yields

$$\zeta(t) = \frac{T}{2}\left(\eta_r + \frac{1}{\eta_r} - \frac{2}{n}\right) \frac{\int_{-\infty}^{t}\left[h^2(0^+,\tau)g^2(0^+,\tau) - h^2(L,\tau)g^2(L,\tau)\right]\,d\tau}{\int_{-\infty}^{\infty} h^2(0^-,t)g^2(0^-,t)dt}, \tag{39}$$

where $T = \kappa^2/\eta_r$ is the transmission coefficient of the vacuum-dielectric interface. The integral in the numerator of 39 is proportional to the instantaneous electromagnetic energy in the slab at a given time t. To express the normalized momentum transfer as a function of time, we choose $t = 0$ to correspond to the instant when the peak of the pulse is located at the vacuum-dielectric interface at $z = 0$ and assume that the pulse duration is significantly shorter than the pulse transit time through the slab, $t_0 = nL/c_0$. The ratio of the integrals in 39 is then zero for $t < 0$ prior to the pulse entering the slab, unity for $0 < t < t_0$ when the pulse is in the slab, and zero for $t > t_0$ after the pulse has left the slab. For an impedance-matched slab where $T = 1$, the normalized momentum transfer is given by

$$\zeta(t) = \begin{cases} 0 & t < 0 \\ 1 - 1/n & 0 < t < t_0 \\ 0 & t > t_0. \end{cases} \tag{40}$$

Thus, the momentum-per-unit-area imparted by the pulse to the slab, $\mathbf{p}_s(t)$, can be expressed as

$$\mathbf{p}_s(t) = \zeta(t)p_0\,\hat{z} = \begin{cases} 0 & t < 0 \\ (1 - 1/n)\,p_0\,\hat{z} & 0 < t < t_0 \\ 0 & t > t_0. \end{cases} \tag{41}$$

We now consider the displacements of the pulse, the slab, and the system as the pulse travels through the slab. Similar to the Balazs thought experiment, we postulate that the pulse carries a mass m. In particular, for a pulse that is uniform over a cross-sectional area A and with dimensions larger than the wavelength, the value of the mass of the pulse in vacuum is given by [6]

$$m = \frac{p_0 A}{c_0} = \frac{E}{c_0^2}, \tag{42}$$

where E is the energy of the pulse. Here, we furthermore assume that this value of pulse mass is independent of location. Due to the absence of loss in the slab, the mass of the slab, M, and the mass of the pulse, m, are invariant for all time. In general, different portions of the slab move with different velocities as the pulse transits through the slab. We simplify our treatment by approximating the slab as a rigid body, with a center-of-mass that is displaced due to the momentum-per-unit-area $p_s(t)$ applied by the pulse. The center-of-mass displacement of the slab, $z_s(t)$, is calculated by integrating $p_s(t)A/M$ over the duration in which the pulse is in the slab to give

$$z_s(t) = \frac{p_s(t)At}{M} = \begin{cases} 0 & t < 0 \\ (1 - 1/n)\, p_0 At/M & 0 < t < t_0 \\ (1 - 1/n)\, p_0 At_0/M & t > t_0. \end{cases} \tag{43}$$

Substituting 42 into 43, we replicate the exact form of the slab center-of-mass displacement calculated by Balazs [6]

$$z_s(t) = \begin{cases} 0 & t < 0 \\ (1 - 1/n)\, mc_0 t/M & 0 < t < t_0 \\ (1 - 1/n)\, mc_0 t_0/M & t > t_0. \end{cases} \tag{44}$$

The center-of-mass displacement of the pulse, $z_p(t)$, is given by

$$z_p(t) = \begin{cases} c_0 t & t < 0 \\ c_0 t/n & 0 < t < t_0 \\ c_0 t - c_0 t_0/n & t > t_0. \end{cases} \tag{45}$$

The center-of-mass displacement of the enclosed slab and pulse system, $z_{sys}(t)$, can be calculated from

$$z_{sys}(t) = \frac{m z_p(t) + M z_s(t)}{m + M}. \tag{46}$$

Substitution of 44 and 45 into 46 yields a system center-of-mass displacement valid for all times

$$z_{sys}(t) = \frac{mc_0 t}{m + M}. \tag{47}$$

The system center-of-mass velocity, v_{sys}, is constant for all time and given by

$$v_{sys} = \frac{\partial z_{sys}}{\partial t} = \frac{mc_0}{m + M}. \tag{48}$$

We next consider the momentum-per-unit-area carried by the pulse at all points in space and time to determine the global momentum of the system. The value of the electromagnetic momentum density in a ponderable medium is not trivially derived and remains a topic of controversy. There are arguments in favor of both an Abraham momentum density [5] given by

$$\mathbf{G}_A = \frac{\mathbf{E} \times \mathbf{H}}{c_0^2} \tag{49}$$

and a Minkowski momentum density [4] given by

$$\mathbf{G}_M = \mathbf{D} \times \mathbf{B}, \tag{50}$$

where $\mathbf{E}$, $\mathbf{H}$, $\mathbf{D}$, and $\mathbf{B}$ are the electric field, magnetic field, electric displacement field, and magnetic flux density field, respectively, in a ponderable medium. For the case of a ponderable medium that is non-dispersive and lossless, the two forms of the momentum density are related by the index of refraction of the medium via

$$\mathbf{G}_M = \epsilon\mu(\mathbf{E} \times \mathbf{H}) = n^2 \frac{\mathbf{E} \times \mathbf{H}}{c_0^2} = n^2 \mathbf{G}_A. \tag{51}$$

First, we postulate that the pulse in the slab possesses an Abraham momentum density given by 49. When the pulse is fully immersed in the slab, the total pulse momentum-per-unit-area at a given time t is obtained by substituting 17 and 18 into 49 and integrating the momentum density, yielding

$$\mathbf{p}_{p,A}(t) = \int_{-\infty}^{\infty} \frac{\mathbf{E}_t(z,t) \times \mathbf{H}_t(z,t)}{c_0^2} dz. \tag{52}$$

We evaluate the total pulse momentum-per-unit-area for full passage of the pulse through a plane at a given position z in the slab by changing the integration variable in 52 to obtain

$$\mathbf{p}_{p,A}(z) = \int_{-\infty}^{\infty} \frac{\mathbf{E}_t(z,t) \times \mathbf{H}_t(z,t)}{c_0^2} \left(\frac{c_0 dt}{n}\right) = \frac{1}{nc_0} \int_{-\infty}^{\infty} \mathbf{E}_t(z,t) \times \mathbf{H}_t(z,t) dt. \tag{53}$$

Substitution of 17 and 18 evaluated at $z = 0^+$ into 53 yields

$$\mathbf{p}_{p,A}(0^+) = \frac{\kappa^2 E_0^2}{nc_0\eta_0} \int_{-\infty}^{\infty} h^2(0^+,t) g^2(0^+,t) dt\, \hat{z}. \tag{54}$$

Given continuity of the electric and magnetic fields at $z = 0$ and an impedance-matched slab where $\kappa^2 = 1$, comparison of 16 and 54 reveals that the pulse momentum-per-unit-area in the slab is modified by a factor of $1/n$ relative to the pulse momentum-per-unit-area in vacuum, consistent with the result of the Balazs thought experiment.

Choosing $t = 0$ to correspond to the instant when the peak of the pulse is located at the vacuum-dielectric interface at $z = 0$ and assuming a pulse duration significantly shorter than the pulse transit time through the slab $t_0 = nL/c_0$, the total pulse momentum-per-unit-area can then be expressed as a function of time as

$$\mathbf{p}_{p,A}(t) = \begin{cases} p_0\, \hat{z} & t < 0 \\ p_0/n\, \hat{z} & 0 < t < t_0 \\ p_0\, \hat{z} & t > t_0 \end{cases} \tag{55}$$

where the pulse is completely immersed in vacuum over the time intervals $t < 0$ and $t > t_0$, and the pulse is completely immersed in the slab over the time interval $0 < t < t_0$.

Adding 41 and 55 yields a conserved total momentum-per-unit-area for all time given by

$$\mathbf{p}_s(t) + \mathbf{p}_{p,A}(t) = p_0. \tag{56}$$

Thus, analytical calculations using Maxwell's equations, the generalized Lorentz force law given in 27, and conservation of slab and pulse mass with a pulse mass given by $m = E/c_0^2$ yield a uniform system center-of-mass velocity (Equations 47 and 48) that corresponds to a starting postulate of the Balazs thought experiment and a slab center-of-mass displacement (Equation 44) that matches one of the intermediate results in the Balazs thought experiment. Further postulating an Abraham form of the electromagnetic momentum density yields a conserved global momentum (Equation 56) at all times that corresponds to another starting postulate of the Balazs thought experiment. The assumptions and postulates of the calculations performed in this section are summarized in Table 2.

We provide an illustrative example by calculating the momentum-per-unit-area and displacement values for a pulse propagating through a lossless slab for a set of parameters detailed in the caption of Fig. 2. As shown in Fig. 2(a), the system momentum-per-unit-area is conserved for all time. Over the time interval in which the pulse is in the slab $0 < t < t_0$, a reduction in the pulse momentum-per-unit-area is balanced by an increase in the slab momentum-per-unit-area. As shown in Fig. 2(b), the slab center-of-mass displacement is initially zero, increases linearly when the pulse is in the slab, and is fixed at a positive value after the pulse has left the slab. The pulse center-of-mass displacement increases with a slope of c_0 for $t < 0$ and $t > t_0$ and a slope of c_0/n for $0 < t < t_0$. The corresponding system center-of-mass displacement is linear and the system center-of-mass velocity is conserved for all time.

Finally, we explore the implication of substituting a Minkowski momentum density (Equation 50) for the Abraham momentum density in the context of the Balazs thought experiment, keeping all other assumptions and postulates the same. Parallelling the treatment for the Abraham momentum density of the pulse (52-54), the total pulse momentum-per-unit-area as a function of time is given by

$$\mathbf{p}_{p,M}(t) = \begin{cases} p_0\,\hat{z} & t < 0 \\ np_0\,\hat{z} & 0 < t < t_0 \\ p_0\,\hat{z} & t > t_0 \end{cases} \tag{57}$$

which is inconsistent with the result of the Balazs thought experiment. The momentum-per-unit-area imparted to the slab by the pulse is independent of any assumption of the pulse momentum density and is once again given by 41. Adding $\mathbf{p}_s(t)$ and $\mathbf{p}_{p,M}(t)$ yields a total momentum-per-unit-area

$$\mathbf{p}_s(t) + \mathbf{p}_{p,M}(t) = \begin{cases} p_0\,\hat{z} & t < 0 \\ (1 - 1/n + n)p_0\,\hat{z} & 0 < t < t_0 \\ p_0\,\hat{z} & t > t_0 \end{cases} \tag{58}$$

which varies as a function of time. Thus, the Minkowski momentum density, in conjunction with Maxwell's equations, the generalized Lorentz force law given in 27 and conservation of slab and pulse mass, do not lead to global momentum conservation at all times. Conservation of global momentum at all times using the Minkowski momentum density may require an alternate form of a generalized Lorentz force law, a different form of the pulse mass in the medium, or both.

3 Numerical Simulations of an Electromagnetic Pulse Incident onto a Lossy Slab

We now consider the case where an electromagnetic pulse is incident onto a slab occupying $0 < z < L$ composed of a non-dispersive, lossy dielectric characterized by a complex relative electric permittivity $\underline{\epsilon}_r$ and complex relative magnetic permeability $\underline{\mu}_r$, with finite imaginary parts, respectively. Our treatment is restricted to the case of a right-handed medium having $\mathrm{Re}[\underline{\epsilon}_r] > 0$ and $\mathrm{Re}[\underline{\mu}_r] > 0$, simultaneously, yielding a complex index of refraction with $\mathrm{Re}[\underline{n}] > 0$. The interaction of the pulse with the slab is described by invoking the four postulates that have been shown, in the previous section, to be consistent with conservation of momentum and conservation of center of mass velocity in the case of a lossless slab: Maxwell's equations, the generalized Lorentz force law given in Equation 27, conservation of both pulse and slab mass, and the Abraham momentum density. Due to the presence of loss in the slab, we invoke an additional postulate to model the transfer of mass from the pulse to the slab as a function of space and time.

An electromagnetic pulse is directed at normal incidence along the $+z$-direction onto the slab. Pulse attenuation in the slab is described by a real, positive electric conductivity, σ_e, and a real, positive magnetic conductivity, σ_m. With the inclusion of loss, Ampere's Law and Faraday's Law (in time-domain notation) become

$$\nabla \times \mathbf{H} = \frac{\partial \epsilon \mathbf{E}}{\partial t} + \sigma_e \mathbf{E}, \tag{59}$$

and

$$\nabla \times \mathbf{E} = -\frac{\partial \mu \mathbf{H}}{\partial t} - \sigma_m \mathbf{H}, \tag{60}$$

respectively. The general parameters are $\epsilon = \epsilon_0$, $\mu = \mu_0$, $\sigma_e = 0$, and $\sigma_m = 0$ in the vacuum region; $\epsilon = \mathrm{Re}[\underline{\epsilon}_r]\epsilon_0$, $\mu = \mathrm{Re}[\underline{\mu}_r]\mu_0$, $\sigma_e = \mathrm{Im}[\underline{\epsilon}_r]\epsilon_0\omega_0$ and $\sigma_m = \mathrm{Im}[\underline{\mu}_r]\mu_0\omega_0$ in the slab region. The spatio-temporal evolution of the pulse is modeled using one-dimensional finite-difference-time-domain FDTD solutions to 59 and 60. The simulation space consists of a one-dimensional array of 4400 pixels with a resolution of 1 nm/pixel. Perfectly-matched-layer boundary conditions are used at the two ends of the simulation space to eliminate spurious reflections from the boundaries. Calculations are performed under two conditions: one in which the slab is impedance-matched to vacuum such that $\underline{\epsilon}_r = \underline{\mu}_r$ and $\eta_r = 1$ and another in which the slab is not impedance-matched to vacuum.

We first perform numerical simulations of electromagnetic pulse propagation through a slab that is impedance-matched to vacuum. The real parts of the relative permittivity and the relative permeability of the slab are both fixed at 2.0. Figures 3(a) and (b) display time-sequences of the FDTD-calculated electric fields for comparative cases in which a pulse is incident onto a lossless slab and a lossy slab, respectively. The incident pulse consists of several electric field oscillations and has a width in vacuum comparable to the width of the slab. Both impedance-matched slabs show no reflection from the either of the two dielectric-vacuum interfaces. The pulse completely transmits through the lossless slab ($\mathrm{Im}[\underline{\epsilon}_r] = 0.0$) with no change in the pulse amplitude and only partially transmits through the lossy slab ($\mathrm{Im}[\underline{\epsilon}_r] = 0.1$) with significant pulse amplitude reduction while the pulse is in the slab.

Based on the electric and magnetic fields obtained from the FDTD simulations, the current densities in the slab are calculated as a function of space and time. The total electric current density consists of a bound electric current density component proportional to $\epsilon - \epsilon_0$ and a free electric current density component proportional to σ_e

$$\mathbf{J}_e(z,t) = \frac{\partial \mathbf{P}(z,t)}{\partial t} + \sigma_e \mathbf{E}(z,t), \tag{61}$$

and the total magnetic current density consists of a bound magnetic current density component proportional to $\mu - \mu_0$ and a free magnetic current density component proportional to σ_m

$$\mathbf{J}_m(z,t) = \frac{\partial \mathbf{M}(z,t)}{\partial t} + \sigma_m \mathbf{H}(z,t). \tag{62}$$

The instantaneous force density, $\mathbf{f}(z,t)$, is obtained by substituting the total current densities given by 61 and 62 into 27 to yield

$$\begin{aligned}
\mathbf{f}(z,t) = &\left(\frac{\partial \mathbf{P}(z,t)}{\partial t} + \sigma_e \mathbf{E}(z,t)\right) \times \mu_0 \mathbf{H}(z,t) - \\
&\left(\frac{\partial \mathbf{M}(z,t)}{\partial t} + \sigma_m \mathbf{H}(z,t)\right) \times \epsilon_0 \mathbf{E}(z,t) \qquad (63)\\
= &\left[(\epsilon - \epsilon_0)\frac{\partial \mathbf{E}(z,t)}{\partial t} + \sigma_e \mathbf{E}(z,t)\right] \times \mu_0 \mathbf{H}(z,t) - \\
&\left[(\mu - \mu_0)\frac{\partial \mathbf{H}(z,t)}{\partial t} + \sigma_m \mathbf{H}(z,t)\right] \times \epsilon_0 \mathbf{E}(z,t) \qquad (64)
\end{aligned}$$

Figure 4 plots a time sequence of the instantaneous Lorentz force density for the case when the pulse is incident onto a lossless slab. The force density is zero throughout the slab prior to the arrival of the pulse. When the pulse traverse the slab, it exerts alternating positive and negative force density over successive quarter cycles of the field, yielding a force density profile that has half the wavelength of the corresponding electric field profile shown in Figure 3(a). The force density returns to zero throughout the slab extent after the pulse has exited the slab.

The instantaneous pressure, $\mathbf{F}(t)$, is obtained by integrating the force density over the extent of the slab

$$\mathbf{F}(t) = \int_0^L \mathbf{f}(z,t)dz. \tag{65}$$

Figure 5(a) plots $\mathbf{F}(t)$ as the pulse traverses a slab for $\mathrm{Im}[\underline{\epsilon}_r]$ values of 0.0, 0.01, 0.1, and 1.0, representative of slabs that are completely transmissive, partially transmissive with slight absorption, partially absorbing with slight transmission, and completely absorbing, respectively. For a completely transmissive slab ($\mathrm{Im}[\underline{\epsilon}_r] = 0.0$), entry of the pulse into the slab over the time interval $-2\,\mathrm{fs} < t < 2\,\mathrm{fs}$ yields an always-positive pressure that is modulated at twice the frequency of the electric field. Alternating positive and negative force densities exerted by the pulse as it enters the slab causes the net pressure exerted on the slab to fluctuate from positive values to zero. When the pulse is fully immersed in the slab over the time interval $2\,\mathrm{fs} < t < 4.2\,\mathrm{fs}$, the positive and negative force densities are balanced, resulting in zero pressure. The exit of the pulse from the slab over the time interval $4.2\,\mathrm{fs} < t < 8.2\,\mathrm{fs}$ yields an always-negative pressure with magnitude and temporal profile exactly matching those of the positive pressure exerted upon entry. A small increase in the absorption of the slab modifies the relative pressures exerted upon pulse entry and exit. In the cases where $\mathrm{Im}[\underline{\epsilon}_r] = 0.01$ and $\mathrm{Im}[\underline{\epsilon}_r] = 0.01$, the positive pressure associated with pulse entry is increased and the negative pressure associated with pulse exit is diminished. The former is attributed to an additional Lorentz force density component due to absorption which is always positive; the latter is attributed simply to the field amplitude reduction when the pulse exits the slab. For the completely absorbing slab ($\mathrm{Im}[\underline{\epsilon}_r] = 1.0$), the pulse exerts only a large positive pressure upon entry and zero pressure after the pulse has been completely absorbed $t > 2\,\mathrm{fs}$.

We calculate $\mathbf{p}_s(t)$ and $\mathbf{p}_{p,A}(t)$ via

$$\mathbf{p}_s(t) = \int_{-\infty}^{t} \mathbf{F}(\tau)d\tau \tag{66}$$

and

$$\mathbf{p}_{p,A}(t) = \int_{-\infty}^{\infty} \mathbf{G}_A(z,t)dz, \tag{67}$$

respectively. Figure 5(b) plots the magnitude of the momentum-per-unit-area imparted by the pulse to the slab, p_s, and the momentum-per-unit-area of the pulse, $p_{p,A}$, for discrete values of $\mathrm{Im}[\underline{\epsilon}_r]$ ranging from 0.0 to 1.0. Before the pulse has arrived at the slab, the momentum of the system is carried entirely by the pulse. As the pulse travels through the slab, changes in p_s are balanced by equal and opposite changes in $p_{p,A}$. For the lossless case, the pulse applies a series of positive pressure spikes upon entry that cause p_s to increase in successive rises and plateaus from 0 to $0.5 \times 10^{-3}\,\mathrm{N\ s/m^2}$ and $p_{p,A}$ to decrease in successive drops and plateaus from $1 \times 10^{-3}\,\mathrm{N\ s/m^2}$ to $0.5 \times 10^{-3}\,\mathrm{N\ s/m^2}$. Both p_s and $p_{p,A}$ are momentarily fixed at $0.5 \times 10^{-3}\,\mathrm{N\ s/m^2}$ after the pulse has fully entered the slab and before the pulse starts to leave the slab. The negative pressure spikes applied by the exiting pulse cause p_s to decrease from $0.5 \times 10^{-3}\,\mathrm{N\ s/m^2}$ to 0 and $p_{p,A}$ to increase from $0.5 \times 10^{-3}\,\mathrm{N\ s/m^2}$ to $1 \times 10^{-3}\,\mathrm{N\ s/m^2}$. For the partially transmissive and partially absorbing cases, increase in p_s relative to that observed for the lossless case is caused by positive Lorentz force associated with absorption in the slab, and decrease in $p_{p,A}$ relative to that observed for the lossless case is caused by diminished pulse amplitude. The portion of the system momentum retained by the slab after the pulse has left the slab increases as the degree of loss in the slab increases; $p_s \simeq 0.22 \times 10^{-3}\,\mathrm{N\ s/m^2}$ for $\mathrm{Im}[\underline{\epsilon}_r] = 0.01$ and $p_s \simeq 0.92 \times 10^{-3}\,\mathrm{N\ s/m^2}$ for $\mathrm{Im}[\underline{\epsilon}_r] = 0.1$. For the completely absorbing slab, p_s increases from 0 to $1.0 \times 10^{-3}\,\mathrm{N\ s/m^2}$ and $p_{p,A}$ decreases from $1.0 \times 10^{-3}\,\mathrm{N\ s/m^2}$ to 0 during the entry of the pulse over the time interval $-2\,\mathrm{fs} < t < 2\,\mathrm{fs}$. For $t > 2\,\mathrm{fs}$, p_s and $p_{p,A}$ remain fixed at $1.0 \times 10^{-3}\,\mathrm{N\ s/m^2}$ and 0, respectively. As shown in Fig. 5(c), the global momentum-per-unit-area of the system is conserved for all time in all cases such that

$$p_s(t) + p_{p,A}(t) = C \tag{68}$$

where C is a constant.

For a pulse that is uniform over a cross-sectional area, A, the mass of the pulse at any time is

$$m(t) = A \int_{-\infty}^{\infty} \rho(z,t) dz \tag{69}$$

where $\rho(z,t)$ is the mass density of the pulse, which is related to the electromagnetic field quantities by [14]

$$\rho(z,t) = \frac{1}{2c_0^2} \left[\mathbf{E}(z,t) \cdot \mathbf{D}(z,t) + \mathbf{H}(z,t) \cdot \mathbf{B}(z,t) \right]. \tag{70}$$

At a given moment in time, the pulse displacement is calculated from

$$z_p(t) = \frac{A \int_{-\infty}^{\infty} \rho(z,t) z dz}{m(t)}. \tag{71}$$

We define m_0 and M_0 to be the initial mass of the pulse and the slab, respectively, before the pulse has entered the slab. Conservation of mass is imposed in the simulations by maintaining a fixed total system mass

$$m_0 + M_0 = m(t) + M(t) \tag{72}$$

where $M(t)$ is the time-dependent mass of the slab. Absorption of the pulse in the slab is modeled by an incremental pulse-mass transfer model. At each time step of the simulation, the incremental change in the mass of the pulse $\Delta m(t+\Delta t) = m(t) - m(t+\Delta t)$ is distributed over the instantaneous normalized mass density profile of the pulse to yield an absorbed mass density over the time increment Δt

$$\rho_a(z, t+\Delta t) = \Delta m(t+\Delta t) \frac{\rho(z, t+\Delta t)}{m(t+\Delta t)}. \tag{73}$$

The absorbed mass density over the time increment Δt is added to the slab mass density at the previous time step, $\rho_s(z,t)$, to yield an updated slab mass density at $t+\Delta t$ given by

$$\rho_s(z, t+\Delta t) = \rho_s(z,t) + \rho_a(z, t+\Delta t) \tag{74}$$

where $M(t+\Delta t) = \int_{-\infty}^{\infty} \rho_s(z, t+\Delta t) dz$. As shown in Fig. 6(a), reductions in the pulse mass are compensated by increases in the slab mass such that the total system mass is conserved for all times. Figure 6(b) plots the total mass density absorbed by the slab for $\mathrm{Im}[\underline{\epsilon}_r]$ values of 0.01, 0.1, and 1.0. As $\mathrm{Im}[\underline{\epsilon}_r]$ increases from 0.01 to 1.0, the peak absorbed mass density increases over two orders of magnitude and the distribution of the absorbed mass density shifts towards the slab interface at $z = 0$. For the completely absorbing slab $\mathrm{Im}[\underline{\epsilon}_r] = 1.0$, the incident pulse is absorbed within $\simeq 0.2\,\mu$m of the interface.

At a given moment in time, the slab center-of-mass displacement is calculated from

$$z_s(t) = \frac{A \int_{-\infty}^{t} p_s(\tau) d\tau + A \int_{-\infty}^{\infty} \rho_s(z,t)(z - L/2) dz}{M(t)}, \tag{75}$$

where the first and second terms in the numerator of 75 describe the shift in the center-of-mass of the slab due to momentum transfer from the pulse to the slab and mass transfer from the pulse to the slab, respectively. Figure 7(a) plots the slab center-of-mass displacement for discrete values of $\mathrm{Im}[\underline{\epsilon}_r]$ ranging from 0.0 to 1.0. For $\mathrm{Im}[\underline{\epsilon}_r] = 0.0$, the pulse applies an accelerating pressure to the slab upon entry and a braking pressure to the slab upon exit, yielding a slab center-of-mass displacement that is initially zero, increases linearly while the pulse propagates through the slab, and remains constant after the pulse exits. For $\mathrm{Im}[\underline{\epsilon}_r] = 0.01$, absorption in the slab reduces the braking pressure exerted by the pulse when it exits the slab such that the slab continues to experience forward displacement even after the pulse exits. For $\mathrm{Im}[\underline{\epsilon}_r] = 0.1$ and $\mathrm{Im}[\underline{\epsilon}_r] = 1.0$, there is significant absorption of the pulse, which deposits the majority of the pulse mass near the interface at $z = 0$ during the entry of the pulse $-1\,\mathrm{fs} < t < 3\,\mathrm{fs}$. Pulse-mass deposition pulls the slab center-of-mass backwards, despite the presence of an always-positive pressure. After initial pull-back of the slab, the positive pressure during pulse entry and weak braking pressure upon pulse exit both contribute to a slab center-of-mass velocity higher than that experienced by the less absorbing slabs.

Based on the slab center-of-mass displacement, we calculate the center-of-mass displacement of the total slab and pulse system from

$$z_{sys}(t) = \frac{m(t)z_p(t) + M(t)z_s(t)}{M(t) + m(t)}. \tag{76}$$

As shown in Fig. 7(b), for the completely transmissive slab, the system center-of-mass moves with a uniform velocity, in agreement with the analytically-derived Equation 48. Although the presence and degree of loss in the slab influence the slab mass density and the slab center-of-mass displacement, they do not influence the system center-of-mass displacement. For slabs with $\mathrm{Im}[\underline{\epsilon}_r] = 1.0$, 0.1, and 0.01, the system center-of-mass moves with a uniform velocity that is equivalent to that observed for the lossless slab for all time. Thus, Maxwell's equations and the Lorentz force law, along with the incremental pulse-mass-transfer model, are consistent with conservation of system center-of-mass velocity, regardless of the degree of loss in the slab.

We next perform numerical simulations of electromagnetic pulse propagation through a slab having $\underline{\epsilon}_r = 3.0 + 0.1i$ and $\underline{\mu}_r = 2.0 + 0.05i$ to demonstrate conserved system center-of-mass velocity and global momentum even in the absence of impedance-matching to vacuum. The pulse consists of a harmonic carrier wave oscillating at a frequency $\omega_0 = 6 \times 10^{14}\,\mathrm{Hz}$ (corresponding to a wavelength $\lambda_0 = 500\,\mathrm{nm}$) and a Gaussian envelope with a full-width-at-half-maximum FWHM= 1 fs. The pulse amplitude is normalized such that the total pulse power is 1 mW. The slab has a length $L = 1\,\mu\mathrm{m}$ and a mass $M = 1\,\mathrm{kg}$.

Figures 8(a) and (b) display time-sequences of the FDTD-calculated electric field and instantaneous Lorentz force density as the pulse is incident onto the slab from vacuum. Due to the non-unity relative impedance of the slab, a small portion of the pulse is reflected when the pulse is incident on either of the two vacuum-dielectric interfaces. The amplitude of the pulse is reduced as it travels through the slab due to absorption. Prior to the arrival of the pulse in the slab, the force density is zero throughout the slab. When the pulse travels through the slab, it exerts alternating positive and negative force density over successive quarter cycles of the field. Absorption in the slab diminishes the force density amplitude when the pulse is in the slab. The force density returns to zero after the pulse has exited the slab (following a sufficient number of multiple reflections back and forth between the internal facets of the slab).

The force-per-unit-area, $\mathbf{F}(t)$, exerted by the pulse onto the slab is plotted in Fig. 9(a). The pulse exerts positive pressure as it enters the slab and negative pressure as it exits the slab. As shown in Fig. 9(b), momentum-per-unit-area in the system is dynamically re-distributed between the slab and the pulse as the pulse travels through the slab, but remains conserved for all time. Prior to the arrival of the pulse, the system momentum is entirely in the pulse and the slab momentum is zero. As the pulse enters the slab, an increase in the slab momentum-per-unit-area is balanced by a commensurate decrease in the pulse momentum-per-unit-area. After the pulse has exited the slab, a greater portion of the system momentum is in the slab and a lesser portion remains in the pulse. As shown in Fig. 9(c), the total momentum-per-unit-area of the system is conserved for all time.

As highlighted in Fig. 10(a), the slab center-of-mass displacement is initially zero and increases approximately linearly while the pulse propagates through and exits the slab. Figure 10(b) show that the corresponding center-of-mass displacement of the total slab and pulse system increases linearly. Thus, even in the absence of impedance matching, an electromagnetic pulse incident onto a lossy slab yields a conserved system center-of-mass velocity for all time. A summary of the assumptions and postulates of the calculations performed in this section are summarized in Table 3.

4 Conclusion

We have re-visited the Balazs thought experiment by using four postulates: Maxwell's equations, a generalized Lorentz force law, conservation of pulse and slab mass, and an Abraham electromagnetic momentum density to describe an electromagnetic pulse propagating through a non-dispersive slab. In contrast to the original Balazs thought experiment, our approach enables investigation of the dynamics of the pulse while it is inside the slab. Pulse interaction with a lossless slab using the four postulates leads to conservation of the system center-of-mass velocity and conservation of global momentum at all times, which correspond to two of the starting postulates of the Balazs thought experiment. In the case where the slab is lossy, we invoke an incremental pulse-mass-transfer model to describe pulse absorption in the slab. Our model of pulse absorption yields conservation of both global momentum and system center-of-mass velocity, regardless of the degree of loss in the slab.

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Appendix A: An Alternative Form of a Generalized Lorentz Force in the Context of the Balazs Thought Experiment

We consider the momentum imparted by an electromagnetic pulse onto a lossless slab using Maxwell's equations and an alternative form of the generalized Lorentz force law expressed as [11–13]

$$\mathbf{f}(z,t) = \mathbf{J}_e(z,t) \times \mathbf{B}(z,t) - \mathbf{J}_m(z,t) \times \mathbf{D}(z,t). \tag{77}$$

Substitution of 20, 21, 24, and 25 into 77 yields the force density

$$\mathbf{f}(z,t) = -\frac{\kappa^2 E_0^2}{n c_0 \eta_r \eta_0} \left(2\epsilon_r \mu_r - \epsilon_r - \mu_r\right) \left(\frac{\partial h}{\partial z} h g^2 + \frac{\partial g}{\partial z} h^2 g \right) \hat{z}. \tag{78}$$

Integrating 78 throughout the extent of the slab yields a total instantaneous force-per-unit-area given by

$$\begin{aligned}\mathbf{F}(t) = {} & \frac{\kappa^2 E_0^2}{2n c_0 \eta_r \eta_0} \left(2\epsilon_r \mu_r - \epsilon_r - \mu_r\right) \\ & \left[h^2(0^+,t) g^2(0^+,t) - h^2(L,t) g^2(L,t) \right] \hat{z}. \end{aligned} \tag{79}$$

Integrating 79 with respect to time yields the net momentum $\mathbf{p}_s(t)$ imparted by the pulse onto the dielectric slab at a given time t

$$\begin{aligned}\mathbf{p}_s(t) = {} & \frac{\kappa^2 E_0^2}{2n c_0 \eta_r \eta_0} \left(2\epsilon_r \mu_r - \epsilon_r - \mu_r\right) \times \\ & \int_{-\infty}^{t} \left[h^2(0^+,\tau) g^2(0^+,\tau) - h^2(L,\tau) g^2(L,\tau) \right] d\tau \, \hat{z}. \end{aligned} \tag{80}$$

The normalized momentum transfer, $\zeta(t)$, is expressed as

$$\zeta(t) = \frac{\kappa^2}{2n\eta_r}\left(2\epsilon_r\mu_r - \epsilon_r - \mu_r\right) \frac{\int_{-\infty}^{t}\left[h^2(0^+,\tau)g^2(0^+,\tau) - h^2(L,\tau)g^2(L,\tau)\right]\,d\tau}{\int_{-\infty}^{\infty} h^2(0^-,t)g^2(0^-,t)dt}. \tag{81}$$

Simplification of the pre-factor in 81 yields

$$\zeta(t) = \frac{T}{2}\left(2n - \eta_r - \frac{1}{\eta_r}\right) \frac{\int_{-\infty}^{t}\left[h^2(0^+,\tau)g^2(0^+,\tau) - h^2(L,\tau)g^2(L,\tau)\right]\,d\tau}{\int_{-\infty}^{\infty} h^2(0^-,t)g^2(0^-,t)dt}. \tag{82}$$

Assuming an impedance-matched slab where $T = 1$ and a pulse duration significantly shorter than $t_0 = nL/c_0$, the normalized momentum transfer is given by

$$\zeta(t) = \begin{cases} 0 & t < 0 \\ n-1 & 0 < t < t_0 \\ 0 & t > t_0 \end{cases} \tag{83}$$

Thus, the momentum-per-unit-area imparted by the pulse to the slab, $\mathbf{p}_s(t)$, is

$$\mathbf{p}_s(t) = \zeta(t)p_0\,\hat{z} = \begin{cases} 0 & t < 0 \\ (n-1)\,p_0\,\hat{z} & 0 < t < t_0 \\ 0 & t > t_0 \end{cases} \tag{84}$$

The required pulse momentum-per-unit-area to achieve conservation of global momentum is

$$\mathbf{p}_p(t) = \begin{cases} p_0\,\hat{z} & t < 0 \\ (2-n)p_0\,\hat{z} & 0 < t < t_0 \\ p_0\,\hat{z} & t > t_0 \end{cases} \tag{85}$$

Because the required magnitude of the pulse momentum-per-unit-area in the dielectric slab, $(2-n)p_0$, is neither physically justifiable (in particular because it would take on negative values for $n > 2$) nor consistent with all previous theories of the electromagnetic momentum density, we conclude that the form of the generalized Lorentz force law given in Equation 77 is inaccurate in describing electromagnetic interactions with ponderable media.

Balazs Thought Experiment

Assumptions	Postulates
– Slab is non-dispersive and lossless with refractive index n – Slab is impedance-matched to vacuum – Slab is massive $M >> m$ – Pulse is a plane wave at normal incidence – Pulse travels with velocity c_0 in vacuum and c_0/n in the slab	– Conservation of total momentum of a closed system – Conservation of system center-of-mass velocity in the absence of external forces – Pulse carries constant mass $m = E/c_0^2$ in vacuum

Table 1 Assumptions and postulates corresponding to the Balazs thought experiment.

Explicit Force Density Calculations: Lossless Case

Assumptions	Postulates
– Slab is non-dispersive and lossless with refractive index n – Slab has a positive index of refraction $n > 0$ – Slab is impedance-matched to vacuum – Slab is massive $M >> m$ – Pulse is a plane wave at normal incidence, with arbitrary envelope and carrier wave – Pulse travels with velocity c_0 in vacuum and c_0/n in the slab – Pulse is short compared to the length of the slab	– Maxwell's equations – Generalized Lorentz force law given in 27 – Electromagnetic momentum density $\mathbf{G}_A = (\mathbf{E} \times \mathbf{H})/c_0^2$ – Slab mass is invariant and pulse carries constant mass $m = E/c_0^2$ independent of its location

Table 2 Assumptions and postulates corresponding to our explicit force density calculations in the case of a pulse incident onto a lossless slab.

Explicit Force Density Calculations: Lossy Case

Assumptions	Postulates
– Slab is non-dispersive and lossy – Slab has a positive index of refraction – Slab has arbitrary impedance mismatch with vacuum – Slab is massive $M >> m$ – Pulse is a plane wave at normal incidence	– Maxwell's equations – Generalized Lorentz force law given in 27 – Electromagnetic momentum density $\mathbf{G}_A = (\mathbf{E} \times \mathbf{H})/c_0^2$ – Spatially-varying electromagnetic mass density $\rho = (\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B})/(2c_0^2)$ – An incremental mass transfer model, proposed here, in which the pulse deposits mass in the slab with a distribution corresponding to the instantaneous mass density profile of the pulse

Table 3 Assumptions and postulates corresponding to our numerical force density calculations in the case of a pulse incident onto a lossy slab.

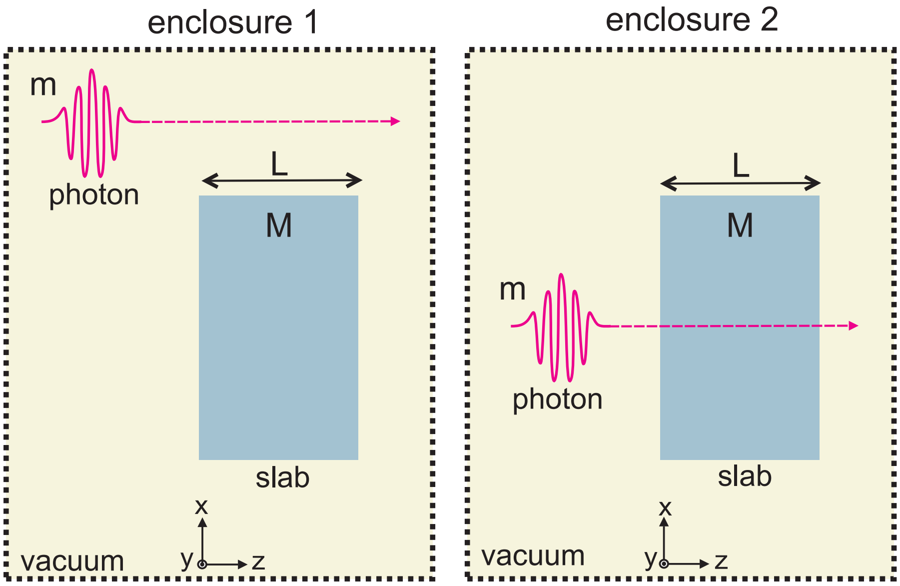
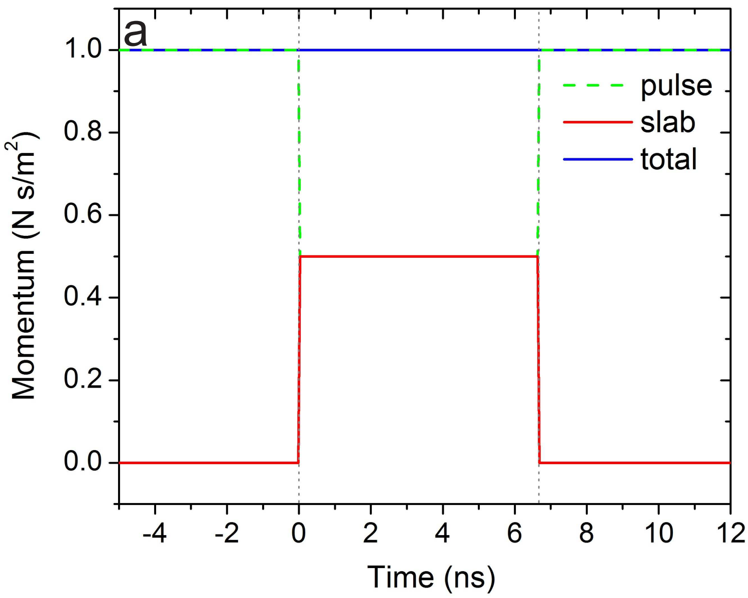


Fig. 1 Formulation of the Balazs thought experiment. Two identical enclosures each contain a photon of mass m and a non-dispersive, lossless slab of mass M and length L. (a) In enclosure 1, the photon propagates in a straight line above the slab through only vacuum. (b) In enclosure 2, the photon propagates in a straight line through both vacuum and the slab.



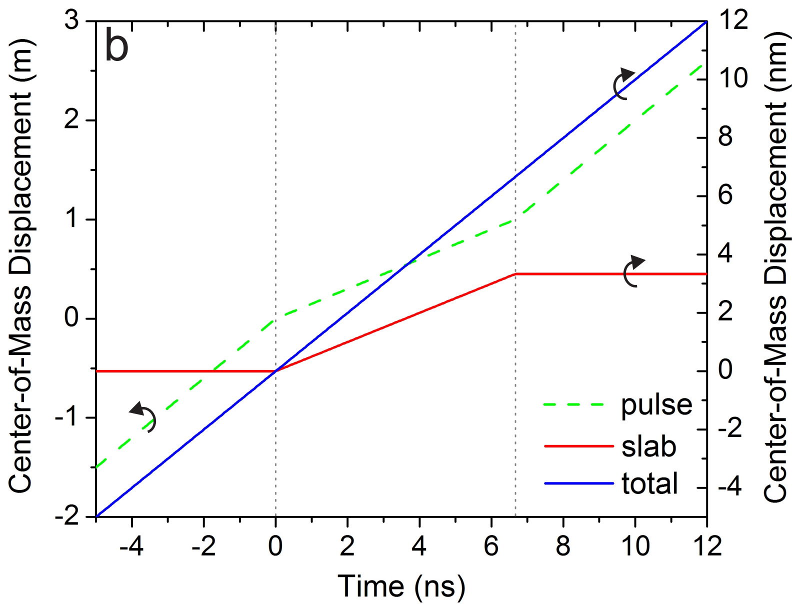


Fig. 2 (a) Momentum-per-unit-area and (b) center-of-mass displacement values calculated for a pulse (green), a non-dispersive, lossless slab (red), and the total pulse-slab system (blue) when the pulse is incident onto the slab. We have assumed a set of parameters $p_0 = 1\,\mathrm{N\ s/m^2}$, $M = 1\,\mathrm{kg}$, $L = 1\mathrm{m}$, $\epsilon_r = 2$, and $\mu_r = 2$. The dotted lines indicate the instances when the peak of the pulse is located at the vacuum-dielectric interfaces at $z = 0$ and $z = L$, respectively.

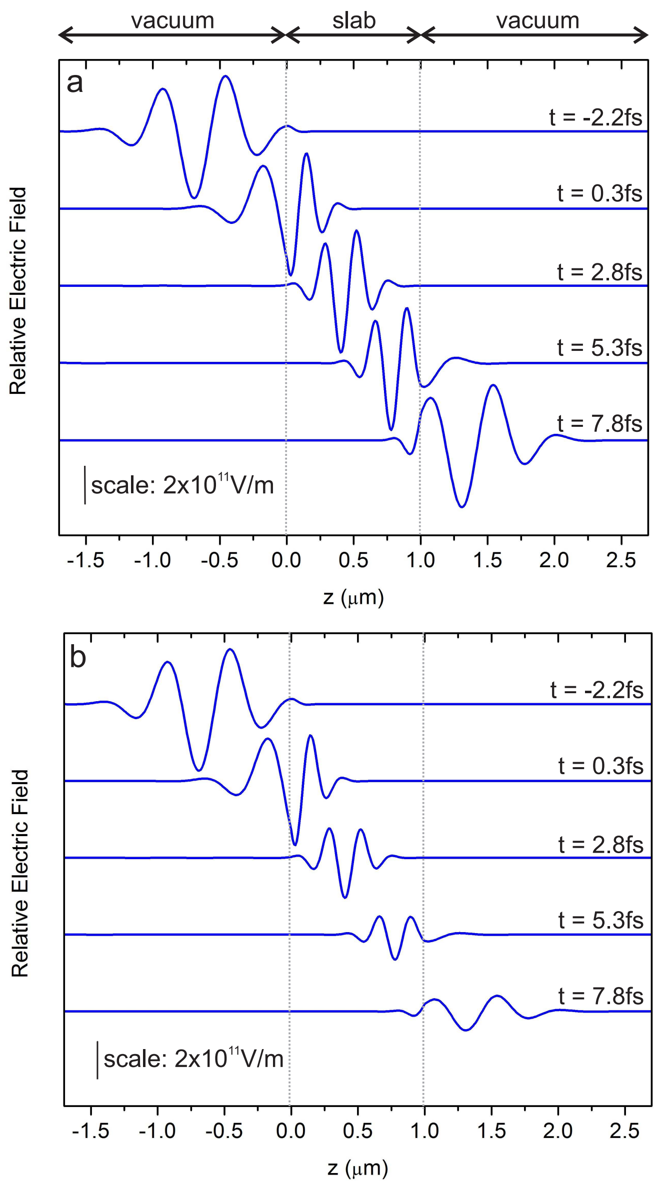


Fig. 3 Time sequence of the FDTD-calculated electric field for (a) a pulse incident onto a non-dispersive, lossless, impedance-matched slab of relative permittivity $\epsilon_r = 2.0$ and (b) a pulse incident onto a lossy, impedance-matched slab of complex relative permittivity $\underline{\epsilon}_r = 2.0 + 0.1i$. For clarity, the curves have been offset such that the horizontal asymptotic value of each curve corresponds to zero electric field. The incident pulse consists of a harmonic carrier wave oscillating at a frequency $\omega_0 = 6 \times 10^{14}$ Hz (corresponding to a wavelength $\lambda_0 = 500$ nm) and a Gaussian envelope with a full-width-at-half-maximum FWHM= 1 fs. The pulse amplitude is normalized such that the total pulse power is 1 mW. The slab has a length $L = 1\,\mu$m and a mass $M = 1$ kg. The dotted lines indicate the edges of the slab.

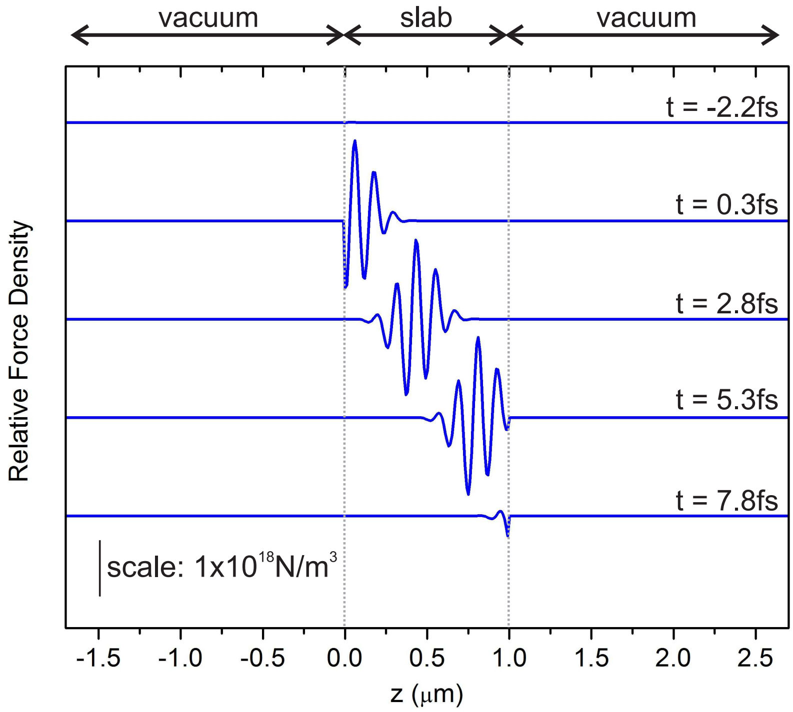


Fig. 4 Time sequence of the FDTD-calculated Lorentz force density for a pulse incident onto a non-dispersive, lossless, impedance-matched slab of relative permittivity $\epsilon_r = 2.0$. For clarity, the curves have been offset such that the horizontal asymptotic values corresponds to zero force density. The dotted lines indicate the edges of the slab.

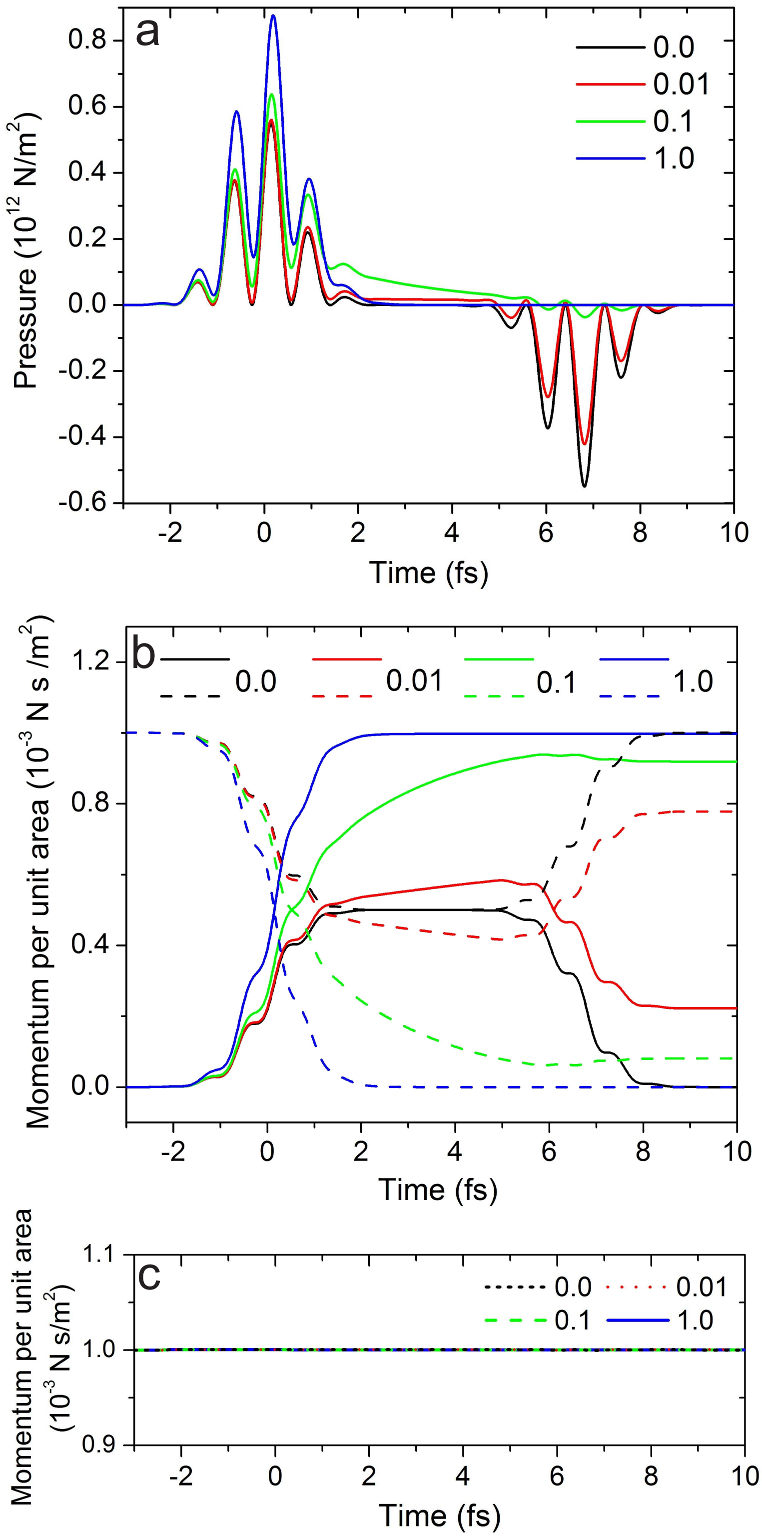


Fig. 5 (a) Instantaneous force-per-unit-area exerted by the pulse onto the slab for discrete $\mathrm{Im}[\epsilon_r]$ values ranging from 0.0 to 1.0. (b) The net momentum-per-unit-area imparted to the slab (solid lines), the momentum-per-unit-area of the pulse (dashed lines), and (c) the total momentum-per-unit-area of the system for discrete $\mathrm{Im}[\epsilon_r]$ values ranging from 0.0 to 1.0.

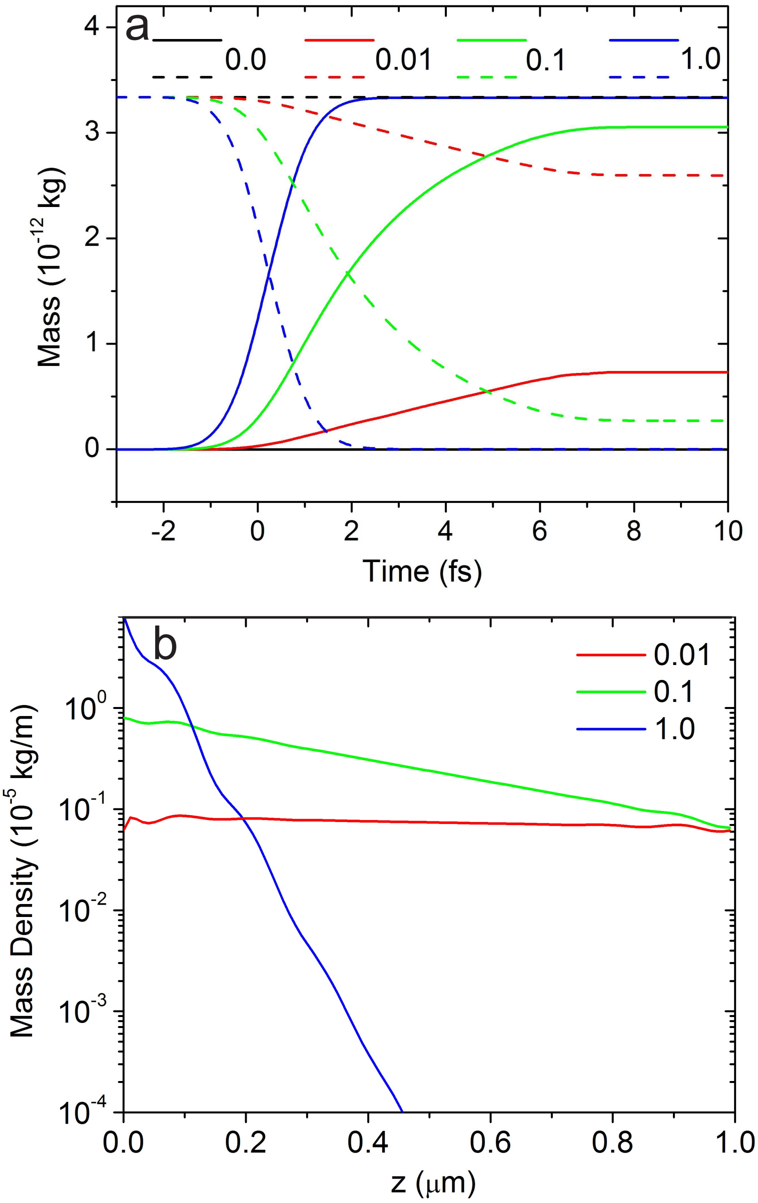


Fig. 6 (a) Total pulse mass (dashed lines) and the change in slab mass (solid lines) as the pulse is incident onto the slab for discrete $\mathrm{Im}[\underline{\epsilon}_r]$ values ranging from 0.0 to 1.0. (b) Mass density absorbed by the slab for discrete $\mathrm{Im}[\underline{\epsilon}_r]$ values ranging from 0.01 to 1.0.

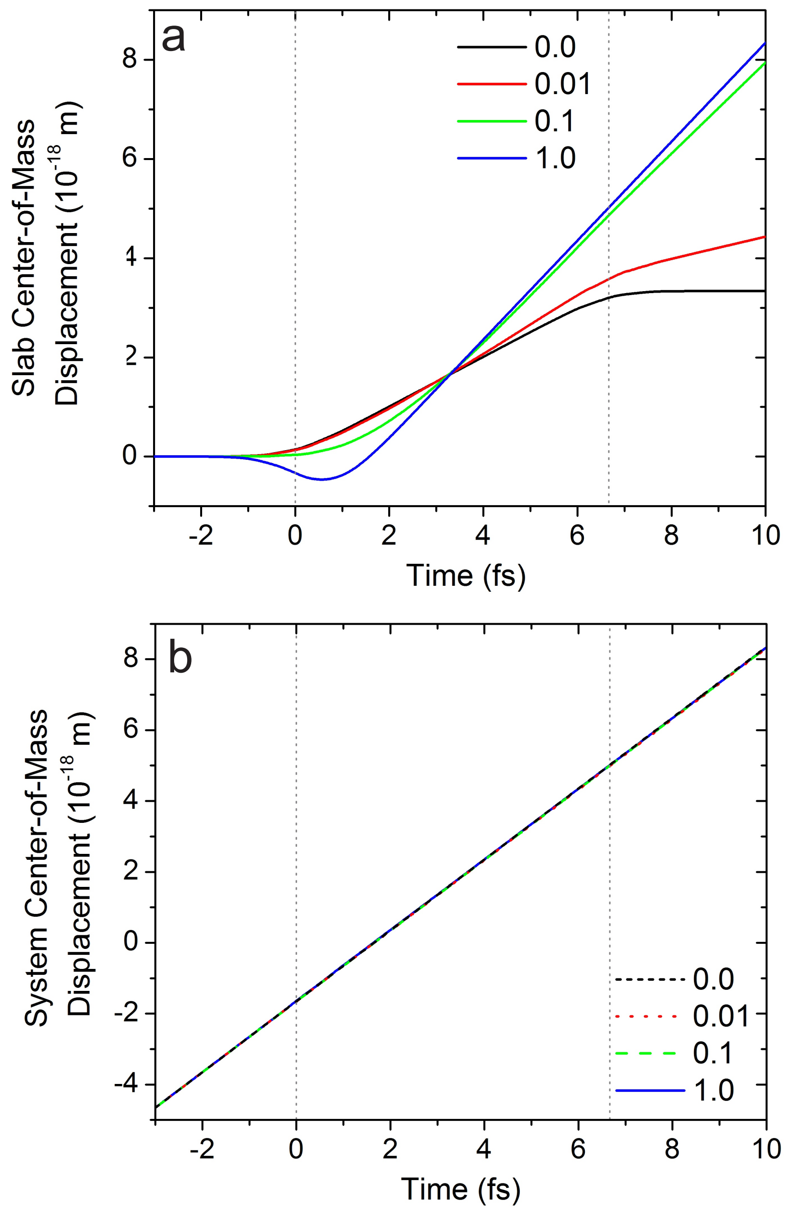


Fig. 7 (a) Slab center-of-mass displacement and (b) system center-of-mass displacement when a pulse traverses a slab for discrete $\mathrm{Im}[\underline{\epsilon}_r]$ values ranging from 0.0 to 1.0. The dotted lines indicate the instances when the peak of the pulse is located at the vacuum-dielectric interfaces at $z = 0$ and $z = L$, respectively, for the case of a non-dispersive, lossless slab. The pulse cross-sectional area is $A = 1\,\mathrm{m}^2$.

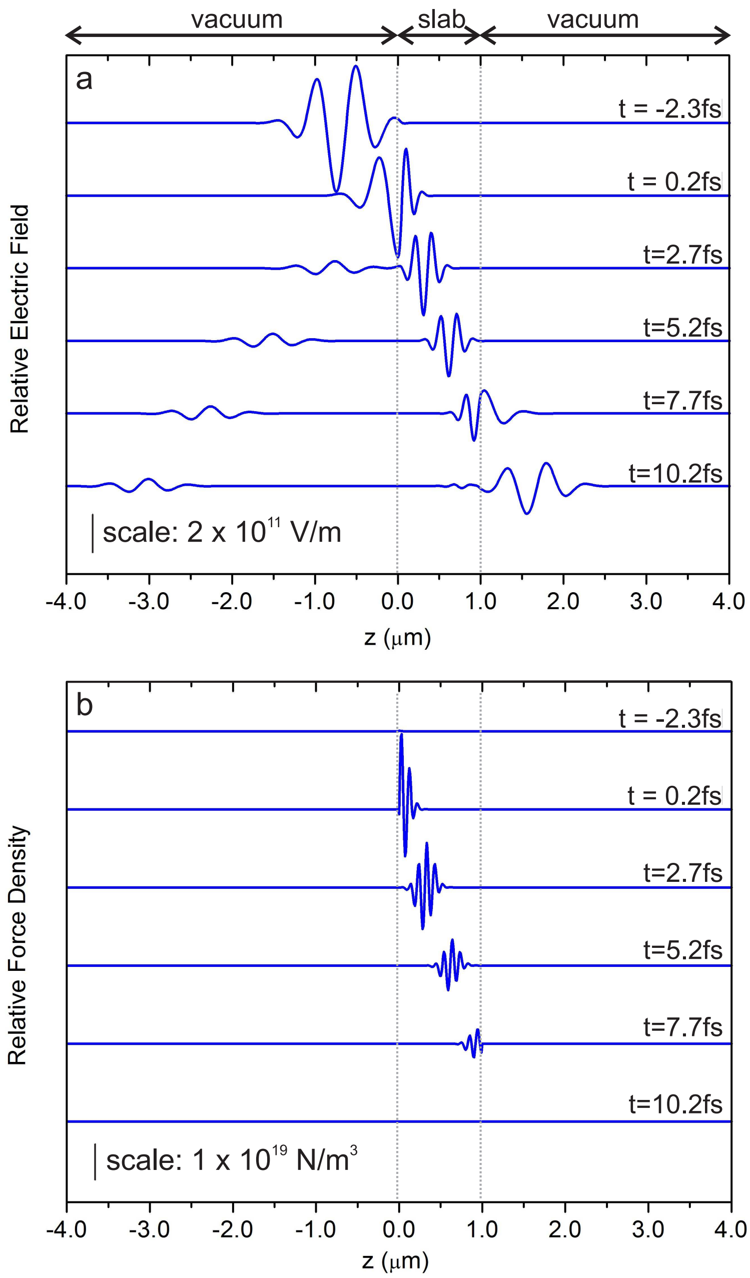


Fig. 8 Time sequence of the FDTD-calculated (a) electric field and (b) Lorentz force density for a pulse incident onto a non-dispersive, lossy slab of complex relative permittivity $\underline{\epsilon}_r = 3.0 + 0.1i$ and complex relative permeability $\underline{\mu}_r = 2.0 + 0.05i$. For clarity, the curves have been vertically offset such that the horizontal asymptotic values for each curve corresponds to zero. The dotted lines indicate the edges of the slab.

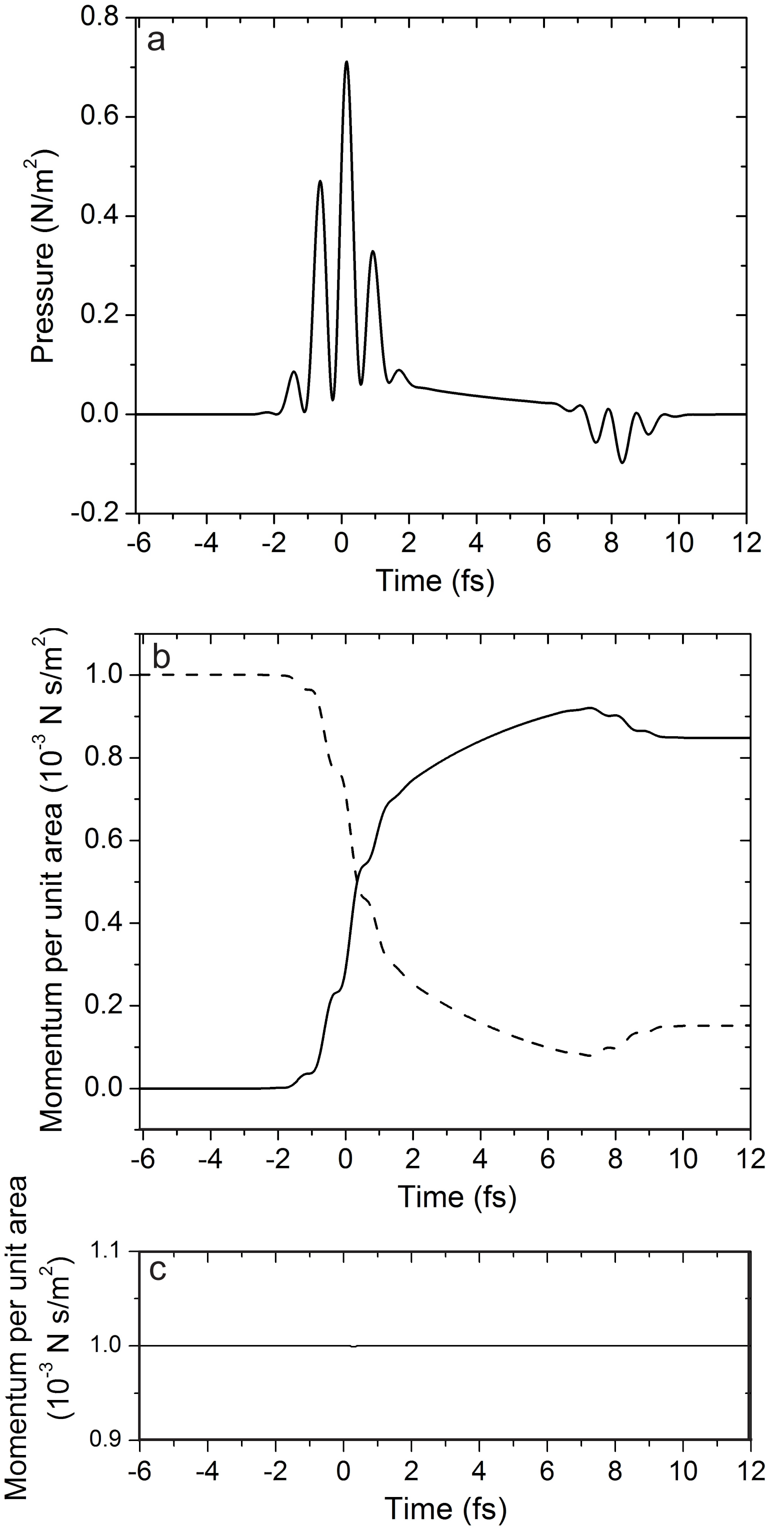


Fig. 9 (a) Instantaneous force-per-unit-area exerted by the pulse onto a non-dispersive, lossy slab of complex relative permittivity $\underline{\epsilon}_r = 3.0 + 0.1i$ and complex relative permeability $\underline{\mu}_r = 2.0 + 0.05i$. (b) The net momentum-per-unit-area imparted to the slab (solid line), the momentum-per-unit-area of the pulse (dashed line), and (c) the total momentum-per-unit-area of the system.

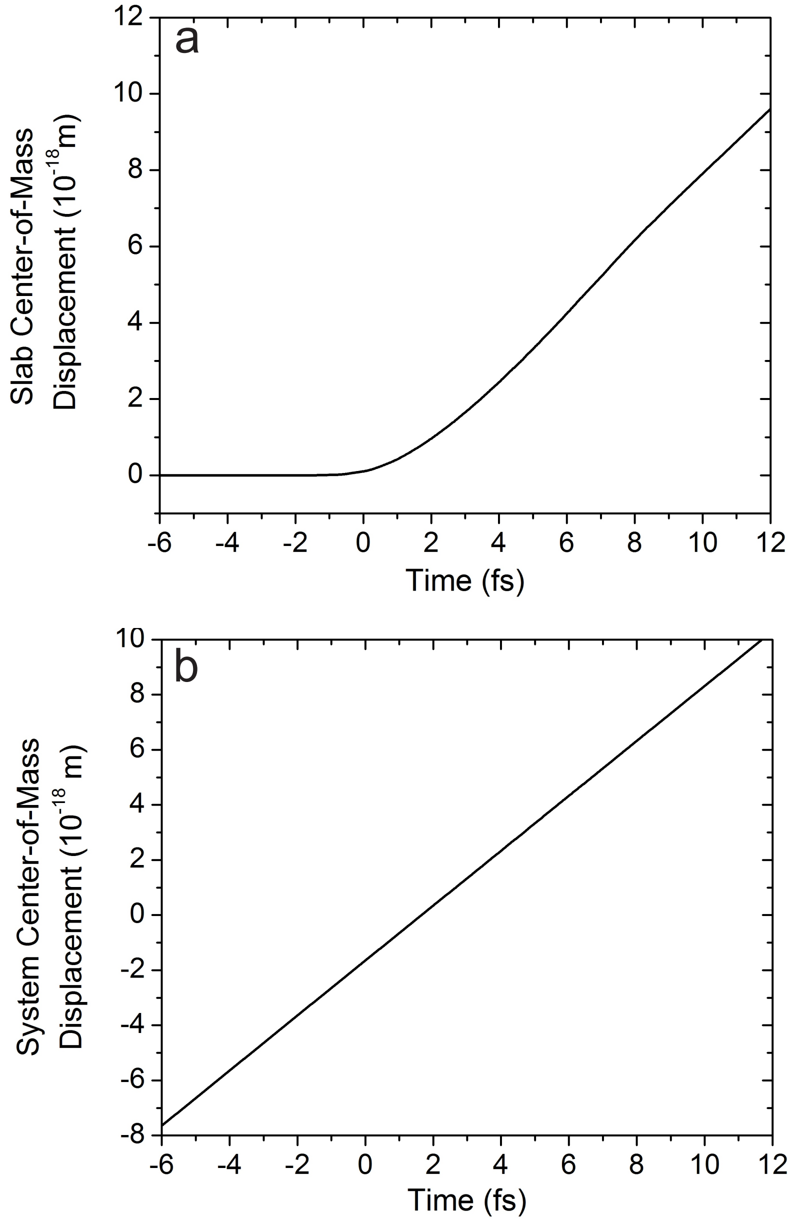


Fig. 10 (a) Slab center-of-mass displacement and (b) system center-of-mass displacement when a pulse traverses a non-dispersive, lossy slab of complex relative permittivity $\underline{\epsilon}_r = 3.0 + 0.1i$ and complex relative permeability $\underline{\mu}_r = 2.0 + 0.05i$. The pulse cross-sectional area is $A = 1\,\mathrm{m}^2$.