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# GENERALIZATIONS AND SPECIALIZATIONS OF GENERATING FUNCTIONS FOR JACOBI, GEGENBAUER, CHEBYSHEV AND LEGENDRE POLYNOMIALS WITH DEFINITE INTEGRALS

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*Abstract.* In this paper we generalize and specialize generating functions for classical orthogonal polynomials, namely Jacobi, Gegenbauer, Chebyshev and Legendre polynomials. We derive a generalization of the generating function for Gegenbauer polynomials through extension a two element sequence of generating functions for Jacobi polynomials. Specializations of generating functions are accomplished through the re-expression of Gauss hypergeometric functions in terms of less general functions. Definite integrals which correspond to the presented orthogonal polynomial series expansions are also given.

## 1. Introduction

This paper concerns itself with analysis of generating functions for Jacobi, Gegenbauer, Chebyshev and Legendre polynomials involving generalization and specialization by re-expression of Gauss hypergeometric generating functions for these orthogonal polynomials. The generalizations that we present here are for two of the most important generating functions for Jacobi polynomials, namely [4, (4.3.1–2)].[1] In fact, these are the first two generating functions which appear in Section 4.3 of [4]. As we will show, these two generating functions, traditionally expressed in terms of Gauss hypergeometric functions, can be re-expressed in terms of associated Legendre functions (and also in terms of Ferrers functions, associated Legendre functions on the real segment $(-1,1)$). Our Jacobi polynomial generating function generalizations, Theorem 1, Corollary 1 and Corollary 2, generalize the generating function for Gegenbauer polynomials. The presented proofs of these generalizations rely upon the series re-arrangment technique. The motivation for the proofs of our generalizations was purely intuitive. Examination of the two Jacobi polynomial generating functions which we generalize,[2] indicate that these generating functions represent two elements of an infinite sequence of eigenfunction expansions.

Our generalized expansions and hypergeometric orthogonal polynomial generating functions are given in terms of Gauss hypergeometric functions. The Gauss hypergeometric Jacobi polynomial generating functions which we generalize, as well as

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[1] The interesting question of which orthogonal polynomial generating functions are important is addressed in [4, Section 4.3] (see also R. A. Askey's MathSciNet review of Srivastava & Manocha (1984) [12]).

[2] As well as their companion identities, see Section 2.


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their eigenfunction expansion generalizations, are all re-expressible in terms of associated Legendre functions. Associated Legendre functions [9, Chapter 14] are given in terms of Gauss hypergeometric functions which satisfy a quadratic transformation of variable. These have an abundance of applications in Physics, Engineering and Applied Mathematics for solving partial differential equations in a variety of contexts. Recently, efficient numerical evaluation of these functions has been investigated in [11]. Associated Legendre functions are (specializations of and) less general than Gauss hypergeometric functions because Gauss hypergeometric functions have three free parameters, whereas associated Legendre functions have only two. One can make this argument of specialization with all of the functions which can be expressed in terms of Gauss hypergeometric functions, namely (inverse) trigonometric, (inverse) hyperbolic, exponential, logarithmic, Jacobi, Gegenbauer, and Chebyshev polynomials, and complete elliptic integrals. We summarize how associated Legendre functions, Gegenbauer, Chebyshev and Legendre polynomials and complete elliptic integrals of the first kind are interrelated. In a one-step process, we obtain definite integrals from our orthogonal polynomial expansions and generating functions.

To the best of our knowledge our generalizations, re-expressions of Gauss hypergeometric generating functions for orthogonal polynomials and definite integrals are new and have not previously appeared in the literature. Furthermore, the generating functions presented in this paper are some of the most important generating functions for these hypergeometric orthogonal polynomials and any specializations and generalizations will be similarly important.

This paper is organized as follows. In Sections 2, 3, 4, 5, 6, we present generalized and specialized expansions for Jacobi, Gegenbauer, Chebyshev of the second kind, Legendre, and Chebyshev of the first kind polynomials respectively. In Appendix A we present definite integrals which correspond to the derived hypergeometric orthgonal polynomial expansions. Unless stated otherwise the domains of convergence given in this paper are those of the original generating function and/or its corresponding definite integral.

Throughout this paper we rely on the following definitions. Let $a_1, a_2, a_3, \ldots \in \mathbf{C}$. If $i, j \in \mathbf{Z}$ and $j < i$, then $\sum_{n=i}^{j} a_n = 0$ and $\prod_{n=i}^{j} a_n = 1$. The set of natural numbers is given by $\mathbf{N} := \{1, 2, 3, \ldots\}$, the set $\mathbf{N}_0 := \{0, 1, 2, \ldots\} = \mathbf{N} \cup \{0\}$, and $\mathbf{Z} := \{0, \pm 1, \pm 2, \ldots\}$. Let $\mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$ be the open unit disk.

## 2. Expansions over Jacobi polynomials

The Jacobi polynomials $P_n^{(\alpha,\beta)} : \mathbf{C} \to \mathbf{C}$ can be defined in terms of a terminating Gauss hypergeometric series as follows ([9, (18.5.7)])

$$P_n^{(\alpha,\beta)}(z) := \frac{(\alpha+1)_n}{n!}\, {}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix}; \frac{1-z}{2}\right),$$

for $n \in \mathbf{N}_0$, and $\alpha, \beta > -1$. The Gauss hypergeometric function ${}_2F_1 : \mathbf{C}^2 \times (\mathbf{C} \setminus \mathbf{N}_0) \times \mathbf{D} \to \mathbf{C}$ (see Chapter 15 in [9]) is defined as

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, $$

where the Pochhammer symbol (rising factorial) $(\cdot)_n : \mathbf{C} \to \mathbf{C}$ [9, (5.2.4)] is defined by

$$ (z)_n := \prod_{i=1}^{n} (z + i - 1), $$

where $n \in \mathbf{N}_0$. Note that the Gauss hypergeometric function can be analytically continued through, for instance, Euler's integral representation for $z \in \mathbf{C} \setminus (1, \infty)$ (see for instance [2, Theorem 2.2.1]).

Consider the generating function for Gegenbauer polynomials (see Section 3 for their definition) given by [9, (18.12.4)], namely

$$ \frac{1}{(1 + \rho^2 - 2\rho x)^{\nu}} = \sum_{n=0}^{\infty} \rho^n C_n^{\nu}(x). \tag{1} $$

We attempt to generalize this expansion using the representation of Gegenbauer polynomials in terms of Jacobi polynomials given by [9, (18.7.1)], namely

$$ C_n^{\nu}(x) = \frac{(2\nu)_n}{\left(\nu + \frac{1}{2}\right)_n} P_n^{(\nu - 1/2, \nu - 1/2)}(x). \tag{2} $$

By making the replacement $\nu - 1/2$ to $\alpha$ and $\beta$ in (1) using (2), we see that there are two possibilities for generalizing the generating function for Gegenbauer polynomials to a generating function for Jacobi polynomials. These two possibilities are given below, namely (3), (7). The first possibility is given for $\rho \in \mathbf{D} \setminus (-1, 0]$ by [9, (18.12.3)]

$$ \begin{aligned} &\frac{1}{(1+\rho)^{\alpha+\beta+1}} {}_2F_1\left(\begin{matrix} \frac{\alpha+\beta+1}{2}, \frac{\alpha+\beta+2}{2} \\ \beta + 1 \end{matrix}; \frac{2\rho(1+x)}{(1+\rho)^2}\right) \\ &\quad = \left(\frac{2}{\rho(1+x)}\right)^{\beta/2} \frac{\Gamma(\beta+1)}{\mathrm{R}^{\alpha+1}} \mathrm{P}_{\alpha}^{-\beta}(\zeta_+) = \sum_{n=0}^{\infty} \frac{(\alpha+\beta+1)_n}{(\beta+1)_n} \rho^n P_n^{(\alpha,\beta)}(x), \end{aligned} \tag{3} $$

where we have used the definitions

$$ \mathrm{R} = \mathrm{R}(\rho, x) := \sqrt{1 + \rho^2 - 2\rho x}, \quad \zeta_{\pm} = \zeta_{\pm}(\rho, x) := \frac{1 \pm \rho}{\sqrt{1 + \rho^2 - 2\rho x}}. $$

Note that the restriction given by $\rho \in \mathbf{D} \setminus (-1, 0]$ is so that the values of $\rho$ are ensured to remain in the domain of $\mathrm{P}_{\nu}^{\mu}$, but may otherwise be relaxed to $\mathbf{D}$ by analytic continuation if one uses the Gauss hypergeometric representation. The Ferrers function of the first kind representation given below provides the analytic continuation to the segment $(-1,0]$. Here $P_\nu^\mu : \mathbf{C}\setminus(-\infty,1]\to\mathbf{C}$ is the associated Legendre function of the first kind (see Chapter 14 in [9]), which can be defined in terms of the Gauss hypergeometric function as follows [9, (14.3.6), (15.2.2), Section 14.21(i)]

$$P_\nu^\mu(z) := \frac{1}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\mu/2} {}_2F_1\left(\begin{array}{c}-\nu,\nu+1\\ 1-\mu\end{array};\frac{1-z}{2}\right). \tag{4}$$

The associated Legendre function of the first kind can also be expressed in terms of the Gauss hypergeometric function as (see [9, (14.3.18), Section 14.21(iii)]), namely

$$P_\nu^\mu(z) = \frac{2^\mu z^{\nu+\mu}}{\Gamma(1-\mu)(z^2-1)^{\mu/2}} {}_2F_1\left(\begin{array}{c}\frac{-\nu-\mu}{2},\frac{-\nu-\mu+1}{2}\\ 1-\mu\end{array};1-\frac{1}{z^2}\right), \tag{5}$$

where $|\arg(z-1)|<\pi$. We have used (5) to re-express the generating function (3). We will refer to a companion identity as one which is produced by applying the map $x\mapsto -x$ to an expansion over Jacobi, Gegenbauer, Chebyshev, or Legendre polynomials with argument $x$, in conjunction with the parity relations for those orthogonal polynomials.

We use the parity relation for Jacobi polynomials (see for instance [9, Table 18.6.1])

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x), \tag{6}$$

and the replacement $\alpha,\beta\mapsto\beta,\alpha$ in (3) producing a companion identity which generalizes the generating function for Gegenbauer polynomials to a generating function for Jacobi polynomials for $\rho\in(0,1)$ by

$$\frac{1}{(1-\rho)^{\alpha+\beta+1}} {}_2F_1\left(\begin{array}{c}\frac{\alpha+\beta+1}{2},\frac{\alpha+\beta+2}{2}\\ \alpha+1\end{array};\frac{-2\rho(1-x)}{(1-\rho)^2}\right)$$
$$= \left(\frac{2}{\rho(1-x)}\right)^{\alpha/2}\frac{\Gamma(\alpha+1)}{\mathrm{R}^{\beta+1}}\mathrm{P}_\beta^{-\alpha}(\zeta_-) = \sum_{n=0}^\infty \frac{(\alpha+\beta+1)_n}{(\alpha+1)_n}\rho^n P_n^{(\alpha,\beta)}(x). \tag{7}$$

Note that the restriction given by $\rho\in(0,1)$ is so that the values of $\rho$ are ensured to remain in the domain of $\mathrm{P}_\nu^\mu$, but may otherwise be relaxed to $\mathbf{D}$ by analytic continuation if one uses the Gauss hypergeometric representation. Here $\mathrm{P}_\nu^\mu:(-1,1)\to\mathbf{C}$ is the Ferrers function of the first kind (associated Legendre function of the first kind on the cut) through [9, (14.3.1)], defined as

$$\mathrm{P}_\nu^\mu(x) := \frac{1}{\Gamma(1-\mu)}\left(\frac{1+x}{1-x}\right)^{\mu/2} {}_2F_1\left(\begin{array}{c}-\nu,\nu+1\\ 1-\mu\end{array};\frac{1-x}{2}\right). \tag{8}$$

The Ferrers function of the first kind can also be expressed in terms of the Gauss hypergeometric function as (see [7, p. 167]), namely

$$\mathrm{P}_\nu^\mu(x) = \frac{2^\mu x^{\nu+\mu}}{\Gamma(1-\mu)(1-x^2)^{\mu/2}} {}_2F_1\left(\begin{array}{c}\frac{-\nu-\mu}{2},\frac{-\nu-\mu+1}{2}\\ 1-\mu\end{array};1-\frac{1}{x^2}\right), \tag{9}$$

for $x\in(0,1)$. We have used (9) to express (7) in terms of the Ferrers function of the first kind. One can easily see that (3) and (7) are generalizations of the generating function for Gegenbauer polynomials by taking $\alpha=\beta=\nu-1/2$. The right-hand sides easily follow using the identification (2) and the left-hand sides follow using [1, (8.6.16–17)].

There exist natural extensions of (3), (7) in the literature (see [4, (4.3.2)]). The extension corresponding to (7) is given for $\rho\in(0,1)$ by

$$\begin{aligned}\frac{(\alpha+\beta+1)(1+\rho)}{(1-\rho)^{\alpha+\beta+2}}{}_2F_1\left(\begin{matrix}\frac{\alpha+\beta+2}{2},\frac{\alpha+\beta+3}{2}\\ \alpha+1\end{matrix};\frac{-2\rho(1-x)}{(1-\rho)^2}\right)\\ =\left(\frac{2}{\rho(1-x)}\right)^{\alpha/2}\frac{(\alpha+\beta+1)(1+\rho)\Gamma(\alpha+1)}{\mathrm{R}^{\beta+2}}\mathrm{P}^{-\alpha}_{\beta+1}(\zeta_-)\\ =\sum_{n=0}^{\infty}(2n+\alpha+\beta+1)\frac{(\alpha+\beta+1)_n}{(\alpha+1)_n}\rho^nP_n^{(\alpha,\beta)}(x),\end{aligned}\tag{10}$$

and its companion identity corresponding to (3), for $\rho\in\mathbf{D}\setminus(-1,0]$ is

$$\begin{aligned}\frac{(\alpha+\beta+1)(1-\rho)}{(1+\rho)^{\alpha+\beta+2}}{}_2F_1\left(\begin{matrix}\frac{\alpha+\beta+2}{2},\frac{\alpha+\beta+3}{2}\\ \beta+1\end{matrix};\frac{2\rho(1+x)}{(1+\rho)^2}\right)\\ =\left(\frac{2}{\rho(1+x)}\right)^{\beta/2}\frac{(\alpha+\beta+1)(1-\rho)\Gamma(\beta+1)}{\mathrm{R}^{\alpha+2}}P^{-\beta}_{\alpha+1}(\zeta_+)\\ =\sum_{n=0}^{\infty}(2n+\alpha+\beta+1)\frac{(\alpha+\beta+1)_n}{(\beta+1)_n}\rho^nP_n^{(\alpha,\beta)}(x).\end{aligned}\tag{11}$$

We have used (5), (9) to re-express these Gauss hypergeometric function generating functions as associated Legendre functions. We have not seen the companion identity (11) in the literature, but it is an obvious consequence of [4, (4.3.2)] using parity. Also, we have not seen the associated Legendre function representations of (10), (11) in the literature.

Upon examination of these two sets of generating functions, we suspected that these were just two examples of an infinite sequence of such expansions. This led us to the proof of the following theorem, which is a Jacobi polynomial expansion which generalizes the generating function for Gegenbauer polynomials (1). According to Ismail (2005) [4, (4.3.2)], the generating functions (10), (11), their generalizations Theorem 1, Corollary 1 and their corresponding definite integrals (45), (45), are closely related to the Poisson kernel for Jacobi polynomials, so our new generalizations will have corresponding applications.

THEOREM 1. *Let* $m\in\mathbf{N}_0$, $\alpha,\beta>-1$, $x\in[-1,1]$, $\rho\in\mathbf{D}\setminus(-1,0]$. *Then*

$$\frac{(1+x)^{-\beta/2}}{\mathrm{R}^{\alpha+m+1}}P^{-\beta}_{\alpha+m}(\zeta_+)=\sum_{n=0}^{\infty}a_{n,m}^{(\alpha,\beta)}(\rho)P_n^{(\alpha,\beta)}(x),\tag{12}$$

*where* $a_{n,m}^{(\alpha,\beta)} : \mathbf{D} \setminus (-1,0] \to \mathbf{C}$ *is defined by*

$$a_{n,m}^{(\alpha,\beta)}(\rho) := \frac{(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)(\alpha+\beta+m+1)_{2n}}{2^{\beta/2}\Gamma(\beta+n+1)} \times \frac{1}{\rho^{(\alpha+1)/2}(1-\rho)^m} P_{-m}^{-\alpha-\beta-2n-1}\left(\frac{1+\rho}{1-\rho}\right).$$

*Proof.* Let $\rho \in (0,\varepsilon)$ with $\varepsilon$ sufficiently small. Then using the definition of the following Gauss hypergeometric function

$${}_2F_1\left(\begin{matrix}\frac{\alpha+\beta+m+1}{2}, \frac{\alpha+\beta+m+2}{2} \\ \beta+1\end{matrix}; \frac{2\rho(1+x)}{(1+\rho)^2}\right) = \sum_{n=0}^{\infty} \frac{\left(\frac{\alpha+\beta+m+1}{2}\right)_n \left(\frac{\alpha+\beta+m+2}{2}\right)_n (2\rho)^n (1+x)^n}{n!(\alpha+1)_n (1+\rho)^{2n}}, \tag{13}$$

the expansion of $(1+x)^n$ in terms of Jacobi polynomials is given by

$$(1+x)^n = 2^n(\beta+1)_n \sum_{k=0}^{n} \frac{(-1)^k(-n)_k(\alpha+\beta+2k+1)(\alpha+\beta+1)_k}{(\alpha+\beta+1)_{n+k+1}(\beta+1)_k} P_k^{(\alpha,\beta)}(x), \tag{14}$$

whose coefficients can be determined using orthogonality of Jacobi polynomials (see Appendix A) combined with the Mellin transform given in [9, (18.17.36)]. By inserting (14) in the right-hand side of (13), we obtain an expansion of the Gauss hypergeometric function on the left-hand side of (13) in terms of Jacobi polynomials. By interchanging the two sums (with justification by absolute convergence), shifting the $n$-index by $k$, and taking advantage of standard properties such as

$$(-n-k)_k = \frac{(-1)^k(n+k)!}{n!},$$

$$(a)_{n+k} = (a)_k(a+k)_n,$$

$$\left(\frac{a}{2}\right)_n \left(\frac{a+1}{2}\right)_n = \frac{1}{2^{2n}}(a)_{2n},$$

$n,k \in \mathbf{N}_0$, $a \in \mathbf{C}$, produces a Gauss hypergeometric function as the coefficient of the Jacobi polynomial expansion. The resulting expansion is

$${}_2F_1\left(\begin{matrix}\frac{\alpha+\beta+m+1}{2}, \frac{\alpha+\beta+m+2}{2} \\ \beta+1\end{matrix}; \frac{2\rho(1+x)}{(1+\rho)^2}\right) = \sum_{n=0}^{\infty} f_{n,m}^{(\alpha,\beta)}(\rho) P_n^{(\alpha,\beta)}(x), \tag{15}$$

where $f_{n,m}^{(\alpha,\beta)} : (0,\varepsilon) \to \mathbf{R}$ is defined by

$$f_{n,m}^{(\alpha,\beta)}(\rho) := \frac{(2n+\alpha+\beta+1)(\alpha+\beta+1)_n(\alpha+\beta+m+1)_{2n}\rho^n}{(\beta+1)_n(\alpha+\beta+1)_{2k+1}(1+\rho)^{2n}}$$
$$\times {}_2F_1\left(\begin{array}{c}\frac{\alpha+\beta+m+2n+1}{2}, \frac{\alpha+\beta+m+2n+2}{2} \\ \alpha+\beta+2n+2\end{array}; \frac{4\rho}{(1+\rho)^2}\right).$$

The above expansion is actually analytic on $\mathbf{D}$. However, if we express it in terms of associated Legendre functions, then we must necessarily subdivide it into two regions. The Gauss hypergeometric function coefficient of this expansion, as well as the Gauss hypergeometric function on the left-hand of (15) are realized to be associated Legendre functions of the first kind through (5). Both sides of the resulting Jacobi polynomial expansion are analytic functions on $\rho \in \mathbf{D}\setminus(-1,0]$. Since we know that (12) is valid for $\rho \in (0,\varepsilon)$, then by the identity theorem for analytic functions, the equation holds on this domain. This completes the proof. $\square$

Note that the left-hand side of Theorem 1 can be rewritten as

$$\frac{(\rho/2)^{\beta/2}}{\Gamma(\beta+1)(1+\rho)^{\alpha+\beta+m+1}}\,{}_2F_1\left(\begin{array}{c}\frac{\alpha+\beta+m+1}{2}, \frac{\alpha+\beta+m+2}{2} \\ \beta+1\end{array}; \frac{2\rho(1+x)}{(1+\rho)^2}\right).$$

We have also derived the companion identity to (12), which we give in the following corollary.

COROLLARY 1. *Let* $m \in \mathbf{N}_0$, $\alpha,\beta > -1$, $x \in [-1,1]$, $\rho \in (0,1)$. *Then*

$$\frac{(1-x)^{-\alpha/2}}{\mathrm{R}^{\beta+m+1}}\mathrm{P}_{\beta+m}^{-\alpha}(\zeta_-) = \sum_{n=0}^{\infty} b_{n,m}^{(\alpha,\beta)}(\rho)P_n^{(\alpha,\beta)}(x), \tag{16}$$

*where* $b_{n,m}^{(\alpha,\beta)} : (0,1) \to \mathbf{R}$ *is defined by*

$$b_{n,m}^{(\alpha,\beta)}(\rho) := \frac{(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)(\alpha+\beta+m+1)_{2n}}{2^{\alpha/2}\Gamma(\alpha+n+1)}$$
$$\times \frac{1}{\rho^{(\beta+1)/2}(1+\rho)^m}\mathrm{P}_{-m}^{-\alpha-\beta-2n-1}\left(\frac{1-\rho}{1+\rho}\right).$$

*Proof.* We start with (15) and apply the parity relation for Jacobi polynomials (6). Let $\rho \in (0,1)$. The Gauss hypergeometric function coefficient of the Jacobi expansion is seen to be a Ferrers function of the first kind (8). After the application of the parity relation, the left-hand side also reduces to a Ferrers function of the first kind through (9). This completes the proof. $\square$

Theorem 1 generalizes (3), (11), while Corollary 1 generalizes (7), (10). Both Theorem 1 and Corollary 1 generalize the generating function for Gegenbauer polynomials (1), which is its own companion identity.

### 2.1. Expansions and definite integrals from the Szegő transformation

If one applies on the complex plane, the Szegő transformation (conformal map)

$$z = \frac{1+\rho^2}{2\rho}, \tag{17}$$

(which maps a circle with radius less than unity to an ellipse with foci at $\pm 1$) to the expansion in Theorem 1, then one obtains a new expansion. By [13, Theorem 12.7.3], this new Jacobi polynomial expansion is convergent for all $x \in \mathbf{C}$ within the interior of this ellipse. Applying (17) to (12) yields the following corollary.

COROLLARY 2. *Let $m \in \mathbf{N}_0$, $\alpha,\beta > -1$, $x,z \in \mathbf{C}$, with $z \in \mathbf{C}\setminus(-\infty,1]$ on any ellipse with the foci at $\pm 1$ and $x$ in the interior of that ellipse. Then*

$$\frac{(1+x)^{-\beta/2}}{(z-x)^{(\alpha+m+1)/2}} P^{-\beta}_{\alpha+m}\left(\frac{1+z-\sqrt{z^2-1}}{\sqrt{2(z-\sqrt{z^2-1})(z-x)}}\right) = \sum_{n=0}^{\infty} c^{(\alpha,\beta)}_{n,m}(z) P^{(\alpha,\beta)}_n(x), \tag{18}$$

*where $c^{(\alpha,\beta)}_{n,m} : \mathbf{C}\setminus(-\infty,1] \to \mathbf{C}$ is defined by*

$$c^{(\alpha,\beta)}_{n,m}(z) := \frac{(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)(\alpha+\beta+m+1)_{2n}}{2^{(\beta-\alpha-m-1)/2}\Gamma(\beta+n+1)} \times \frac{(z-\sqrt{z^2-1})^{m/2}}{\left(1-z+\sqrt{z^2-1}\right)^m} P^{-\alpha-\beta-2n-1}_{-m}\left(\sqrt{\frac{z+1}{z-1}}\right).$$

We would just like to briefly note that one may use the Szegő transformation (17) to obtain new expansion formulae and corresponding definite integrals from all the Jacobi, Gegenbauer, Legendre and Chebyshev polynomial expansions used in this paper. For the sake of brevity, we leave this to the reader.

## 3. Expansions over Gegenbauer polynomials

The Gegenbauer polynomials $C^{\mu}_n : \mathbf{C} \to \mathbf{C}$ can be defined in terms of a terminating Gauss hypergeometric series as follows ([9, (18.5.9)])

$$C^{\mu}_n(z) := \frac{(2\mu)_n}{n!}\, {}_2F_1\left(\begin{matrix} -n, n+2\mu \\ \mu+\frac{1}{2} \end{matrix}; \frac{1-z}{2}\right), \tag{19}$$

for $n \in \mathbf{N}_0$ and $\mu \in (-1/2,\infty)\setminus\{0\}$. The Gegenbauer polynomials (19) are defined for $\mu \in (-1/2,\infty)\setminus\{0\}$. However many of the formulae listed below actually make sense in the limit as $\mu \to 0$. In this case, one should take the limit of the expression as $\mu \to 0$ with the interpretation of obtaining Chebyshev polynomials of the first kind (see Section 6 for the details of this limiting procedure).

COROLLARY 3. *Let* $m \in \mathbf{N}_0$, $\mu \in (-1/2, \infty) \setminus \{0\}$, $x \in [-1,1]$. *If* $\rho \in \mathbf{D} \setminus (-1,0]$, *then*

$$\frac{1}{\mathrm{R}^{2\mu+m}} C_m^{\mu}(\zeta_+) = \frac{2\Gamma(2\mu+m)}{m!\rho^{\mu}(1-\rho)^m} \sum_{n=0}^{\infty} (n+\mu)(2\mu+m)_{2n} P_{-m}^{-2\mu-2n}\left(\frac{1+\rho}{1-\rho}\right) C_n^{\mu}(x), \tag{20}$$

*and if* $\rho \in (0,1)$ *then*

$$\frac{1}{\mathrm{R}^{2\mu+m}} C_m^{\mu}(\zeta_-) = \frac{2\Gamma(2\mu+m)}{m!\rho^{\mu}(1+\rho)^m} \sum_{n=0}^{\infty} (n+\mu)(2\mu+m)_{2n} \mathrm{P}_{-m}^{-2\mu-2n}\left(\frac{1-\rho}{1+\rho}\right) C_n^{\mu}(x). \tag{21}$$

*Proof.* Using (12), substitute $\alpha = \beta = \mu - 1/2$ along with (2) and [9, (14.3.22)], namely

$$P_{n+\mu-1/2}^{1/2-\mu}(z) = \frac{2^{\mu-1/2}\Gamma(\mu)n!}{\sqrt{\pi}\,\Gamma(2\mu+n)}(z^2-1)^{\mu/2-1/4} C_n^{\mu}(z).$$

Through (4), we see that the Gauss hypergeometric function in the definition of the associated Legendre function of the first kind on the right-hand side is terminating and therefore defines an analytic function for $\rho \in \mathbf{D}$. The analytic continuation to the segment $\rho \in (0,1]$ is provided by replacing the associated Legendre function of the first kind with the Ferrers function of the first kind with argument $(1-\rho)/(1+\rho)$. □

As an example for re-expression using specialization to associated Legendre functions which was mentioned in the introduction, we now apply to two generating function results of Koekoek *et al.* (2010) [6, (9.8.32)] and Rainville (1960) [10, (144.8)].

THEOREM 2. *Let* $\lambda \in \mathbf{C}$, $\mu \in (-1/2, \infty) \setminus \{0\}$, $\rho \in (0,1)$, $x \in [-1,1]$. *Then*

$$(1-x^2)^{1/4-\mu/2} P_{\mu-\lambda-1/2}^{1/2-\mu}(\mathrm{R}+\rho)\, \mathrm{P}_{\mu-\lambda-1/2}^{1/2-\mu}(\mathrm{R}-\rho)$$
$$= \frac{2^{1/2-\mu}}{\Gamma(\mu+\frac{1}{2})} \sum_{n=0}^{\infty} \frac{(\lambda)_n (2\mu-\lambda)_n}{(2\mu)_n \Gamma(\mu+\frac{1}{2}+n)} \rho^{\mu-1/2+n} C_n^{\mu}(x). \tag{22}$$

*Proof.* The formula [6, (9.8.32)] gives a generating function for Gegenbauer polynomials, namely

$${}_2F_1\left(\begin{matrix}\lambda, 2\mu-\lambda \\ \mu+\frac{1}{2}\end{matrix}; \frac{1-\mathrm{R}-\rho}{2}\right) {}_2F_1\left(\begin{matrix}\lambda, 2\mu-\lambda \\ \mu+\frac{1}{2}\end{matrix}; \frac{1-\mathrm{R}+\rho}{2}\right)$$
$$= \sum_{n=0}^{\infty} \frac{(\lambda)_n (2\mu-\lambda)_n}{(2\mu)_n (\mu+\frac{1}{2})_n} \rho^n C_n^{\mu}(x).$$

Using [9, (15.8.17)] to apply a quadratic transformation to the Gauss hypergeometric functions and then using (5), with degree and order given by $\mu-\lambda-1/2$, $1/2-\mu$, respectively and either $z = \mathrm{R}+\rho$ or $z = \mathrm{R}-\rho$, with simplification completes the proof. □

THEOREM 3. *Let* $\alpha \in \mathbf{C}$, $\mu \in (-1/2,\infty)\setminus\{0\}$, $\rho \in (0,1)$, $x \in [-1,1]$. *Then*

$$\frac{\left(1-x^2\right)^{1/4-\mu/2}}{\mathrm{R}^{1/2+\alpha-\mu}}\mathrm{P}^{1/2-\mu}_{\mu-\alpha-1/2}\left(\frac{1-\rho x}{\mathrm{R}}\right) = \frac{(\rho/2)^{\mu-1/2}}{\Gamma(\mu+\frac{1}{2})}\sum_{n=0}^{\infty}\frac{(\alpha)_n}{(2\mu)_n}\rho^n C_n^{\mu}(x). \tag{23}$$

*Proof.* On p. 279 of [10, (144.8)] there is a generating function for Gegenbauer polynomials, namely

$$(1-\rho x)^{-\alpha}\,{}_2F_1\left(\begin{matrix}\frac{\alpha}{2},\frac{\alpha+1}{2}\\ \mu+\frac{1}{2}\end{matrix};\frac{-\rho^2(1-x^2)}{(1-\rho x)^2}\right) = \sum_{n=0}^{\infty}\frac{(\alpha)_n}{(2\mu)_n}\rho^n C_n^{\mu}(x).$$

Using (5) to rewrite the Gauss hypergeometric function on the left-hand side of the above equation completes the proof. □

## 4. Expansions over Chebyshev polynomials of the second kind

The Chebyshev polynomials of the second kind can be obtained from the Gegenbauer polynomials using [9, (18.7.4)], namely

$$U_n(z) = C_n^1(z), \tag{24}$$

for $n \in \mathbf{N}_0$. Hence and through (19), the Chebyshev polynomials of the second kind $U_n : \mathbf{C} \to \mathbf{C}$ can be defined in terms of a terminating Gauss hypergeometric series as follows

$$U_n(z) := (n+1)\,{}_2F_1\left(\begin{matrix}-n,n+2\\ \frac{3}{2}\end{matrix};\frac{1-z}{2}\right). \tag{25}$$

COROLLARY 4. *Let* $m \in \mathbf{N}_0$, $x \in [-1,1]$. *If* $\rho \in \mathbf{D}\setminus(-1,0]$ *then*

$$\frac{1}{\mathrm{R}^{m+2}}U_m(\zeta_+) = \frac{2(m+1)}{\rho(1-\rho)^m}\sum_{n=0}^{\infty}(n+1)(m+2)_{2n}P^{-2n-2}_{-m}\left(\frac{1+\rho}{1-\rho}\right)U_n(x), \tag{26}$$

*and if* $\rho \in (0,1)$ *then*

$$\frac{1}{\mathrm{R}^{m+2}}U_m(\zeta_-) = \frac{2(m+1)}{\rho(1+\rho)^m}\sum_{n=0}^{\infty}(n+1)(m+2)_{2n}\mathrm{P}^{-2n-2}_{-m}\left(\frac{1-\rho}{1+\rho}\right)U_n(x). \tag{27}$$

*Proof.* Using (20), with (24) and

$$U_m(z) = \sqrt{\frac{\pi}{2}}\frac{m+1}{(z^2-1)^{1/4}}P^{-1/2}_{m+1/2}(z), \tag{28}$$

which follows from (4), (25), and [9, (15.8.1)]. Analytically continuing to the segment $\rho \in (0,1)$ completes the proof. □

Note that using [1, (8.6.9)], namely

$$P_\nu^{-1/2}(z) = \sqrt{\frac{2}{\pi}} \frac{(z^2-1)^{-1/4}}{(2\nu+1)} \left[ \left(z+\sqrt{z^2-1}\right)^{\nu+1/2} - \left(z+\sqrt{z^2-1}\right)^{-\nu-1/2} \right],$$

and (28) one can derive the elementary function representation for the Chebyshev polynomials of the second kind [8, (1.52)].

COROLLARY 5. *Let* $\rho \in \mathbf{D}$, $x \in [-1,1]$. *Then*

$$(1-x^2)^{-1/4} P_{1/2-\lambda}^{-1/2}(\mathrm{R}+\rho) \mathrm{P}_{1/2-\lambda}^{-1/2}(\mathrm{R}-\rho)$$
$$= \frac{2^{5/2}\sqrt{\rho}}{\pi} \sum_{n=0}^{\infty} \frac{(\lambda)_n (2-\lambda)_n 2^{2n} \rho^n}{(2n+2)!} U_n(x). \tag{29}$$

*Proof.* Substituting $\mu = 1$ into (22), and using (24) with simplification, completes the proof. $\square$

COROLLARY 6. *Let* $\alpha \in \mathbf{C}$, $\rho \in (0,1)$, $x \in [-1,1]$. *Then*

$$\frac{\mathrm{R}^{1/2-\alpha}}{(1-x^2)^{1/4}} \mathrm{P}_{1/2-\alpha}^{-1/2}\left(\frac{1-\rho x}{\mathrm{R}}\right) = \sqrt{\frac{2\rho}{\pi}} \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(n+1)!} \rho^n U_n(x). \tag{30}$$

*Proof.* Using (24), and substituting $\mu = 1$ into (23) with simplification, produces this generating function for Chebyshev polynomials of the second kind. $\square$

## 5. Expansions over Legendre polynomials

Legendre polynomials can be obtained from the Gegenbauer polynomials using [9, (18.7.9)], namely

$$P_n(z) = C_n^{1/2}(z), \tag{31}$$

for $n \in \mathbf{N}_0$. Hence and through (19), the Legendre polynomials $P_n : \mathbf{C} \to \mathbf{C}$ can be defined in terms of a terminating Gauss hypergeometric series as follows

$$P_n(z) := {}_2F_1\left(\begin{matrix} -n, n+1 \\ 1 \end{matrix}; \frac{1-z}{2}\right). \tag{32}$$

Using (31) we can write the previous expansions over Gegenbauer polynomials in terms of expansions over Legendre polynomials.

COROLLARY 7. *Let* $m \in \mathbf{N}_0$, $x \in [-1,1]$. *If* $\rho \in \mathbf{D} \setminus (-1,0]$ *then*

$$\frac{1}{\mathrm{R}^{m+1}} P_m(\zeta_+) = \frac{(1-\rho)^{-m}}{\sqrt{\rho}} \sum_{n=0}^{\infty} (2n+1)(m+1)_{2n} P_{-m}^{-2n-1}\left(\frac{1+\rho}{1-\rho}\right) P_n(x), \tag{33}$$

*and if* $\rho \in (0,1)$ *then*

$$\frac{1}{\mathrm{R}^{m+1}} P_m(\zeta_-) = \frac{(1+\rho)^{-m}}{\sqrt{\rho}} \sum_{n=0}^{\infty} (2n+1)(m+1)_{2n} \mathrm{P}_{-m}^{-2n-1}\left(\frac{1-\rho}{1+\rho}\right) P_n(x). \tag{34}$$

*Proof.* Using (20), substitute $\mu = 1/2$ with (31) and $P_m(z) = P_m^0(z)$, which follows from (4), (32). Analytic continuation to $\rho \in (0,1)$ completes the proof. □

COROLLARY 8. *Let* $\lambda \in \mathbf{C}$, $\rho \in \{x \in \mathbf{C} : |z| < 1\}$, $x \in [-1,1]$. *Then*

$$P_{-\lambda}(\mathrm{R}+\rho)\mathrm{P}_{-\lambda}(\mathrm{R}-\rho) = \sum_{n=0}^{\infty} \frac{(\lambda)_n (1-\lambda)_n}{(n!)^2} \rho^n P_n(x). \tag{35}$$

*Proof.* Substituting $\mu = 1/2$ into (22) and using (31) with simplification completes the proof. □

Note that Corollary 8 is just a restatement of [14, Theorem A], and therefore Theorem 2 is a generalization of Brafman's theorem.

COROLLARY 9. *Let* $\alpha \in \mathbf{C}$, $\rho \in \mathbf{D}$, $x \in [-1,1]$. *Then*

$$\mathrm{R}^{-\alpha} \mathrm{P}_{\alpha-1}\left(\frac{1-\rho x}{\mathrm{R}}\right) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} \rho^n P_n(x). \tag{36}$$

*Proof.* Substituting $\mu = 1/2$ in (23) with simplification completes the proof. □

As a further example of the specialization to associated Legendre functions mentioned in the introduction, we apply to the recent generating function results of Wan & Zudelin (2012) [14].

THEOREM 4. *Let* $x, y$ *be in a neighborhood of 1. Then*

$$\frac{\pi^2}{2} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{(n!)^2} P_{2n}\left(\frac{(x+y)(1-xy)}{(x-y)(1+xy)}\right)\left(\frac{x-y}{1+xy}\right)^{2n}$$

$$= \begin{cases} \dfrac{\pi^2}{2} & \text{if } x = y = 1, \\[2ex] \dfrac{1+xy}{xy} K\left(\dfrac{\sqrt{x^2-1}}{x}\right) K\left(\dfrac{\sqrt{y^2-1}}{y}\right) & \text{if } x, y \geqslant 1, \\[2ex] \dfrac{1+xy}{x} K\left(\dfrac{\sqrt{x^2-1}}{x}\right) K\left(\sqrt{1-y^2}\right) & \text{if } x \geqslant 1 \text{ and } y \leqslant 1, \\[2ex] \dfrac{1+xy}{y} K\left(\sqrt{1-x^2}\right) K\left(\dfrac{\sqrt{y^2-1}}{y}\right) & \text{if } x \leqslant 1 \text{ and } y \geqslant 1, \\[2ex] (1+xy) K\left(\sqrt{1-x^2}\right) K\left(\sqrt{1-y^2}\right) & \text{if } x, y \leqslant 1. \end{cases}$$

*Proof.* If we start with (10) from [14], namely

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2}{(n!)^2} P_{2n}\left(\frac{(x+y)(1-xy)}{(x-y)(1+xy)}\right)\left(\frac{x-y}{1+xy}\right)^{2n}$$
$$= \frac{1+xy}{2}\,{}_2F_1\left(\begin{matrix}\frac{1}{2},\frac{1}{2}\\ 1\end{matrix};1-x^2\right){}_2F_1\left(\begin{matrix}\frac{1}{2},\frac{1}{2}\\ 1\end{matrix};1-y^2\right),$$

and use [9, (15.9.21)] we can express the Gauss hypergeometric functions as Legendre functions. For instance

$${}_2F_1\left(\begin{matrix}\frac{1}{2},\frac{1}{2}\\ 1\end{matrix};1-x^2\right) = P_{-1/2}(2x^2-1),$$

with $x \in \mathbf{C}\setminus(-\infty,0]$. This domain is as such because the Legendre function of the first kind $P_\nu$ and the Ferrers function of the first kind $\mathrm{P}_\nu$, both with order $\mu = 0$, are given by the same Gauss hypergeometric function and are continuous across argument unity (cf. (4), (8)). So there is no distinction between these two functions, except that the Ferrers function has argument on the real line with modulus less than unity and the Legendre function is defined on $\mathbf{C}\setminus(-\infty,1)$ (both being well defined with argument unity). (Hence there really is no need to use two different symbols to denote this function.) The proof is completed by noting the two formulae

$$P_{-1/2}(z) = \frac{2}{\pi}\sqrt{\frac{2}{z+1}}K\left(\sqrt{\frac{z-1}{z+1}}\right), \qquad \mathrm{P}_{-1/2}(x) = \frac{2}{\pi}K\left(\sqrt{\frac{1-x}{2}}\right)$$

[1, (8.13.1), (8.13.8)], where $K : [0,1) \to [\pi/2,\infty)$ is the complete elliptic integral of the first kind defined by [9, (19.2.8)]

$$K(k) := \frac{\pi}{2}\,{}_2F_1\left(\begin{matrix}\frac{1}{2},\frac{1}{2}\\ 1\end{matrix};k^2\right). \qquad \square$$

THEOREM 5. *Let $x,y$ be in a neighborhood of 1. Then*

$$3\sum_{n=0}^{\infty} \frac{\left(\frac{1}{3}\right)_n\left(\frac{2}{3}\right)_n}{(n!)^2} P_{3n}\left(\frac{x+y-2x^2y^2}{(x-y)\sqrt{1+4xy(x+y)}}\right)\left(\frac{x-y}{\sqrt{1+4xy(x+y)}}\right)^{3n}.$$

$$= \sqrt{1+4xy(x+y)}\begin{cases} 1 & \text{if } x=y=1\\ P_{-1/3}\left(2x^3-1\right)P_{-1/3}\left(2y^3-1\right) & \text{if } x,y>1\\ \mathrm{P}_{-1/3}\left(2x^3-1\right)P_{-1/3}\left(2y^3-1\right) & \text{if } x<1 \text{ and } y>1\\ P_{-1/3}\left(2x^3-1\right)\mathrm{P}_{-1/3}\left(2y^3-1\right) & \text{if } x>1 \text{ and } y<1\\ \mathrm{P}_{-1/3}\left(2x^3-1\right)\mathrm{P}_{-1/3}\left(2y^3-1\right) & \text{if } x,y<1 \end{cases}$$

*Proof.* This follows by [14, (11)] and [9, (15.9.21)]. $\square$

## 6. Expansions over Chebyshev polynomials of the first kind

The Chebyshev polynomials of the first kind $T_n:\mathbf{C}\to\mathbf{C}$ can be defined in terms of a terminating Gauss hypergeometric series as follows [7, p. 257]

$$T_n(z) := {}_2F_1\left(\begin{matrix}-n,n\\ \frac{1}{2}\end{matrix};\frac{1-z}{2}\right), \tag{37}$$

for $n\in\mathbf{N}_0$. The Chebyshev polynomials of the first kind can be obtained from the Gegenbauer polynomials using [2, (6.4.13)], namely

$$T_n(z) = \frac{1}{\varepsilon_n}\lim_{\mu\to 0}\frac{n+\mu}{\mu}C_n^\mu(z), \tag{38}$$

where the Neumann factor $\varepsilon_n\in\{1,2\}$, commonly seen in Fourier cosine series, is defined as $\varepsilon_n := 2-\delta_{n,0}$.

COROLLARY 10. *Let* $m\in\mathbf{N}_0,\ x\in[-1,1]$. *If* $\rho\in\mathbf{D}\setminus(-1,0]$ *then*

$$\frac{1}{\mathrm{R}^m}T_m(\zeta_+) = \frac{1}{(1-\rho)^m}\sum_{n=0}^\infty \varepsilon_n(m)_{2n}P_{-m}^{-2n}\left(\frac{1+\rho}{1-\rho}\right)T_n(x), \tag{39}$$

*and if* $\rho\in(0,1)$ *then*

$$\frac{1}{\mathrm{R}^m}T_m(\zeta_-) = \frac{1}{(1+\rho)^m}\sum_{n=0}^\infty \varepsilon_n(m)_{2n}\mathrm{P}_{-m}^{-2n}\left(\frac{1-\rho}{1+\rho}\right)T_n(x). \tag{40}$$

*Proof.* Using (20), (38), and

$$T_m(z) = \sqrt{\frac{\pi}{2}}(z^2-1)^{1/4}P_{m-1/2}^{1/2}(z), \tag{41}$$

which follows from (4), (37), [9, (15.8.1)]. Analytic continuation to $\rho\in(0,1)$ completes the proof. □

Note that using [1, (8.6.8)], namely

$$P_\nu^{1/2}(z) = \frac{1}{\sqrt{2\pi}}\left(z^2-1\right)^{-1/4}\left[\left(z+\sqrt{z^2-1}\right)^{\nu+1/2}+\left(z+\sqrt{z^2-1}\right)^{-\nu-1/2}\right],$$

and (41) one can derive the elementary function representation for the Chebyshev polynomials of the first kind [5, p. 177].

## Appendix A. Definite integrals

As a consequence of the series expansions given above, one may generate corresponding definite integrals (in a one-step procedure) as an application of the orthogonality relation for these hypergeometric orthogonal polynomials. We now describe this correspondence. Given an expansion over a set of orthogonal polynomials $p_n$ such that

$$f(x)=\sum_{n=0}^{\infty}a_np_n(x), \tag{42}$$

and the orthogonality relation

$$\int_{-1}^{1}p_n(x)p_m(x)w(x)dx=c_n\delta_{n,m}, \tag{43}$$

where $w:(-1,1)\to[0,\infty)$, then using (43) one has

$$\int_{-1}^{1}f(x)p_n(x)w(x)dx=\sum_{m=0}^{\infty}a_m\int_{-1}^{1}p_n(x)p_m(x)w(x)dx=a_nc_n,$$

and therefore

$$a_n=\frac{1}{c_n}\int_{-1}^{1}f(x)p_n(x)w(x)dx. \tag{44}$$

The definite integral expression (44) for the coefficient $a_n$ is of equal importance to the expansion (42), since one may use it to derive the other. Integrals of such sort are always of interest since they are very likely to find applications in applied mathematics and theoretical physics and could be included in tables of integrals such as [3].

For Jacobi, Gegenbauer, Chebyshev of the second kind, Legendre, and Chebyshev of the first kind polynomials, the orthogonality relations can be found in [9, (18.2.1), (18.2.5), Table 18.3.1]. Using the above procedure, we obtain the following definite integrals for products of special functions with Jacobi, Gegenbauer, Chebyshev and Legendre polynomials. Let $m,n\in\mathbf{N}_0$, $\alpha,\beta>-1$, $\rho\in\{z\in\mathbf{C}:0<|z|<1\}\setminus(-1,0]$. Then

$$\int_{-1}^{1}\frac{(1-x)^{\alpha}(1+x)^{\beta/2}}{\mathrm{R}^{\alpha+m+1}}P_{\alpha+m}^{-\beta}(\zeta_+)P_n^{(\alpha,\beta)}(x)dx$$
$$=\frac{2^{\alpha+\beta/2+1}\Gamma(\alpha+n+1)(\alpha+\beta+m+1)_{2n}}{n!\rho^{(\alpha+1)/2}(1-\rho)^m}P_{-m}^{-\alpha-\beta-2n-1}\left(\frac{1+\rho}{1-\rho}\right).$$

Let $\rho\in(0,1)$. Then

$$\int_{-1}^{1}\frac{(1-x)^{\alpha/2}(1+x)^{\beta}}{\mathrm{R}^{\beta+m+1}}\mathrm{P}_{\beta+m}^{-\alpha}(\zeta_-)P_n^{(\alpha,\beta)}(x)dx$$
$$=\frac{2^{\alpha/2+\beta+1}\Gamma(\beta+n+1)(\alpha+\beta+m+1)_{2n}}{n!\rho^{(\beta+1)/2}(1+\rho)^m}\mathrm{P}_{-m}^{-\alpha-\beta-2n-1}\left(\frac{1-\rho}{1+\rho}\right).$$

Let $\mu\in(-1/2,\infty)\setminus\{0\}$, $\rho\in\{z\in\mathbf{C}:0<|z|<1\}\setminus(-1,0]$. Then

$$\int_{-1}^{1}\frac{(1-x^2)^{\mu-1/2}}{\mathrm{R}^{2\mu+m}}C_m^\mu(\zeta_+)C_n^\mu(x)dx = \frac{2^{2-2\mu}\pi\Gamma(2\mu+n)\Gamma(2\mu+2n+m)}{m!n!\Gamma^2(\mu)\rho^\mu(1-\rho)^m}P_{-m}^{-2n-2\mu}\left(\frac{1+\rho}{1-\rho}\right).$$

A similar integral on $\rho\in(0,1)$ can be obtained using (21).

It should be noted that by using (24), (31), (38), the previous definite integral over Gegenbauer polynomials can be written as an integral over Chebyshev polynomials of the first and second kind, and well as Legendre polynomials. Let $\lambda\in\mathbf{C}$, $\rho\in\mathbf{D}$. Then

$$\int_{-1}^{1}(1-x^2)^{\mu/2-1/4}P_{\mu-1/2-\lambda}^{1/2-\mu}(\mathrm{R}+\rho)\,\mathrm{P}_{\mu-1/2-\lambda}^{1/2-\mu}(\mathrm{R}-\rho)\,C_n^\mu(x)dx = \frac{(\lambda)_n(2\mu-\lambda)_n\rho^{n+\mu-1/2}2^{\mu-1/2}}{(n+\mu)\Gamma(2\mu)(\mu+1/2)_n n!}.$$

Let $\rho\in(0,1)$. Then

$$\int_{-1}^{1}\frac{(1-x^2)^{1/4-\mu/2}}{\mathrm{R}^{1/2-\mu+\lambda}}\mathrm{P}_{\mu-\lambda-1/2}^{1/2-\mu}\left(\frac{1-\rho x}{\mathrm{R}}\right)C_n^\mu(x)dx = \frac{(\lambda)_n\sqrt{\pi}\rho^{n+\mu-1/2}2^{1/2-\mu}}{(n+\mu)\Gamma(\mu)n!}.$$

The previous two definite integrals over Gegenbauer polynomials can also be written as integrals over Chebyshev polynomials of the second kind and Legendre polynomials using (24), (31).

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