

# Constructive Relationships Between Algebraic Thickness and Normality<sup>\*</sup>

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**Abstract.** We study the relationship between two measures of Boolean functions; *algebraic thickness* and *normality*. For a function  $f$ , the algebraic thickness is a variant of the *sparsity*, the number of nonzero coefficients in the unique  $\mathbb{F}_2$  polynomial representing  $f$ , and the normality is the largest dimension of an affine subspace on which  $f$  is constant. We show that for  $0 < \epsilon < 2$ , any function with algebraic thickness  $n^{3-\epsilon}$  is constant on some affine subspace of dimension  $\Omega\left(n^{\frac{\epsilon}{2}}\right)$ . Furthermore, we give an algorithm for finding such a subspace. This is at most a factor of  $\Theta(\sqrt{n})$  from the best guaranteed, and when restricted to the technique used, is at most a factor of  $\Theta(\sqrt{\log n})$  from the best guaranteed. We also show that a concrete function, majority, has algebraic thickness  $\Omega\left(2^{n^{1/6}}\right)$ .

## 1 Introduction and Known Results

Boolean functions play an important role in many areas of computer science. In cryptology, Boolean functions are sometimes classified according to measures of complexity (also called cryptographic complexity [7], nonlinearity criteria [17] or nonlinearity measures [1]). Examples of such measures are *nonlinearity*, *algebraic degree*, *normality*, *algebraic thickness* and *multiplicative complexity*, and there are a number of results showing that functions having a small value according to a certain measure are vulnerable to a certain attack (see [8] for a good survey).

A significant amount of work in this area presents explicit functions that achieve high (or low) values according to some measure. For the *nonlinearity* measure this was settled by showing the existence of bent functions [21], for *algebraic degree* the problem is trivial, for *multiplicative*

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<sup>\*</sup> Partially supported by the Danish Council for Independent Research, Natural Sciences.

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*complexity* this is a well studied problem in circuit complexity [3], for *normality* this is exactly the problem of finding good *affine dispersers* [22]. The first result in this paper is that the majority function has exponential algebraic thickness.

Another line of work has been to establish relationships between these measures, e.g. considering questions of the form “if a function  $f$  is simple (or complex) according to one measure, what does that say about  $f$  according to some other measure”, see e.g. [4, 8] and the references therein. In this paper we focus on the relationship between *algebraic thickness* and *normality*. Intuitively, these measures capture, each in their own way, how “far” functions are from being linear [6, 7]. In fact, these two measures have been studied together previously (see e.g. [5, 6]). The relationship between these measures was considered in the work of Cohen and Tal in [10], where they show that functions with a certain algebraic thickness have a certain normality. For relatively small values of algebraic thickness, we tighten their bounds and present an algorithm to witness this normality. The question of giving a constructive proof of normality is not just a theoretical one. Recently a generic attack on stream ciphers with low normality was successfully mounted in the work [19]. If it is possible to constructively compute a witness of normality given a function with low algebraic thickness, this implies that any function with low algebraic thickness is likely to be vulnerable to the attack in [19], as well as any other attack based on normality. This work suggests that this is indeed possible for functions with small algebraic thickness.

## 2 Preliminaries and Known Results

Let  $\mathbb{F}_2$  be the field of order 2,  $\mathbb{F}_2^n$  the  $n$ -dimensional vector space over  $\mathbb{F}_2$ , and  $[n] = \{1, \dots, n\}$ . A mapping from  $\mathbb{F}_2^n$  to  $\mathbb{F}_2$  is called a *Boolean function*. It is a well known fact that any Boolean function  $f$  in the variables  $x_1, \dots, x_n$  can be expressed uniquely as a multilinear polynomial over  $\mathbb{F}_2$  called the *algebraic normal form* (ANF) or the *Zhegalkin polynomial*. That is, there exist unique constants  $c_\emptyset, \dots, c_{\{1, \dots, n\}}$ , such that

$$f(x_1, \dots, x_n) = \sum_{S \subseteq [n]} c_S \prod_{j \in S} x_j,$$

where arithmetic is in  $\mathbb{F}_2$ . In the rest of this paper, most arithmetic will be in  $\mathbb{F}_2$ , although we still need arithmetic in  $\mathbb{R}$ . If nothing is mentioned it should be clear from the context what field is referred to. The largest  $|S|$  such that  $c_S = 1$  is called the (*algebraic*) *degree* of  $f$ , and functions

with degree 2 are called *quadratic* functions. We let  $\log$  be the logarithm base two,  $\ln$  the natural logarithm, and  $\exp(\cdot)$  the natural exponential function with base  $e$ .

*Algebraic Thickness* For a Boolean function,  $f$ , let  $\|f\| = \sum_{S \subseteq [n]} c_S$ , with arithmetic in  $\mathbb{R}$ . This measure is sometimes called the *sparsity* of  $f$  (e.g. [10]). The *algebraic thickness* [4, 6] of  $f$ , denoted  $\mathcal{T}(f)$  is defined as the smallest sparsity after any affine bijection has been applied to the inputs of  $f$ . More precisely, letting  $\mathcal{A}_n$  denote the set of affine, bijective operators on  $\mathbb{F}_2^n$ ,

$$\mathcal{T}(f) = \min_{A \in \mathcal{A}_n} \|f \circ A\|. \quad (1)$$

Algebraic thickness was introduced and first studied by Carlet in [4–6]. Affine functions have algebraic thickness at most 1, and Carlet showed that for any constant  $c > \sqrt{\ln 2}$ , for sufficiently large  $n$  there exist functions with algebraic thickness  $2^{n-1} - cn2^{\frac{n-1}{2}}$ , and that a *random* Boolean function will have such high algebraic thickness with high probability. Furthermore *no* function has algebraic thickness larger than  $\frac{2}{3}2^n$ . Carlet observes that algebraic thickness was also implicitly mentioned in [18, Page 208] and related to the so called “higher order differential attack” due to Knudsen [15] and Lai [16] in that they are dependent on the degree as well as the number of terms in the ANF of the function used.

*Normality* A  $k$ -dimensional *flat* is an affine (sub)space of  $\mathbb{F}_2^n$  with dimension  $k$ . A function is  $k$ -*normal* if there exists a  $k$ -dimensional flat  $E$  such that  $f$  is constant on  $E$  [9, 4]. For simplicity define the *normality* of a function  $f$ , which we denote  $\mathcal{N}(f)$ , as the *largest*  $k$  such that  $f$  is  $k$ -normal. We recall that affine functions have normality at least  $n - 1$ , while for any  $c > 1$ , a random Boolean function has normality less than  $c \log n$  with high probability.

Functions with normality smaller than  $k$  are often called *affine dispersers* of dimension  $k$ , and a great deal of work has been put into explicit constructions of functions with low normality. Currently the asymptotically best known deterministic function, due to Shaltiel, has normality less than  $2^{\log^{0.9} n}$  [22].

Notice the asymmetry: linear functions have very low algebraic thickness ( $\leq 1$ ) but very high normality, ( $\geq n - 1$ ), whereas random functions, with high probability, have very high algebraic thickness ( $\geq 2^{n-1} - 0.92 \cdot n \cdot 2^{\frac{n-1}{2}}$ ) but low normality ( $\leq 1.01 \log n$ ) [5].

*Remark on Computational Efficiency* In this paper, we say that something is efficiently computable if it is computable in time polynomially bounded in the size of the input. Algorithms in this paper will have a Boolean function with a certain algebraic thickness as input. We assume that a function  $f$  with algebraic thickness  $\mathcal{T}(f) = T$  is represented by a function  $g$  and an affine bijection  $A$  such that  $g = f \circ A$  and  $\|g\| = T$ . In this setting, representing a function  $f$  uses  $\text{poly}(\mathcal{T}(f) + n^2)$  bits.

*Quadratic Functions* The normality and algebraic thickness of quadratic functions are well understood due to the following theorem due to Dickson [11] (see also [8] for a proof).

**Theorem 1 (Dickson).** *Let  $f: \mathbb{F}_2^n \mapsto \mathbb{F}_2$  be quadratic. Then there exist an invertible  $n \times n$  matrix  $A$ , a vector  $\mathbf{b} \in \mathbb{F}_2^n$ ,  $t \leq \frac{n}{2}$ , and  $c \in \mathbb{F}_2$  such that for  $\mathbf{y} = A\mathbf{x} + \mathbf{b}$  one of the following two equations holds:*

$$f(x) = y_1y_2 + y_3y_4 + \dots y_{t-1}y_t + c, \text{ or } f(x) = y_1y_2 + y_3y_4 + \dots y_{t-1}y_t + y_{t+1}.$$

Furthermore  $A$ ,  $\mathbf{b}$  and  $c$  can be found efficiently.

That is, any quadratic function is affine equivalent to some inner product function. We highlight a simple but useful consequence of Theorem 1. Simply by setting one variable in each of the degree two terms to zero, one gets:

**Proposition 1.** *Let  $f: \mathbb{F}_2^n \rightarrow \mathbb{F}_2$  be quadratic. Then  $\mathcal{N}(f) \geq \lfloor \frac{n}{2} \rfloor$ . Furthermore a flat witnessing the normality of  $f$  can be found efficiently.*

*Some Relationships* It was shown in [6] that normality and algebraic thickness are logically independent of (that is, not subsumed by) each other. Several other results relating algebraic thickness and normality to other cryptographic measures are given in [6]. We mention a few relations to other measures.

Clearly, functions with degree  $d$  have algebraic thickness  $O(n^d)$ , so having superpolynomial algebraic thickness requires superconstant degree. The fact that there exist functions with low degree and low normality has been established in [4] and [10] independently. In the following, by a *random degree three polynomial*, we mean a function where each term of degree three is included in the ANF independently with probability  $\frac{1}{2}$ . No other terms are included in the ANF.

**Theorem 2** ([4, 10]<sup>3</sup>). *Let  $f$  on  $n$  variables be a random degree three polynomial. Then with high probability,  $f$  remains nonconstant on any subspace of dimension  $6.12\sqrt{n}$ .*

In fact, as mentioned in [10], for any constant  $d$ , a random degree  $d$  polynomial has normality  $O(n^{1/(d-1)})$ . Perhaps surprisingly, this is tight. More precisely the authors give an elegant proof showing that *any* function with degree  $d$  has  $\mathcal{N}(f) \in \Omega(n^{1/(d-1)})$ . This result implies the following relation between algebraic thickness and normality.

**Theorem 3 (Cohen and Tal [10]).** *Let  $c$  be an integer and let  $f$  have  $\mathcal{T}(f) \leq n^c$ . Then  $\mathcal{N}(f) \in \Omega(n^{1/(4c)})$ .*

The proof of this has two steps: First they show by probabilistic methods that  $f$  has a restriction with a certain number of free variables and a certain degree, and after this they appeal to a relation between degree and normality. Although the authors do study the algorithmic question of finding such a subspace, they do not propose an efficient algorithm for finding a subspace of such dimension. We will pay special attention to the following type of restrictions of Boolean functions.

**Definition 1.** *Let  $f: \mathbb{F}_2^n \rightarrow \mathbb{F}_2$ . Setting  $k < n$  of the bits to 0 results in a function  $f'$  on  $n - k$  variables. We say that  $f'$  is a 0-restriction of  $f$ .*

By inspecting the proof in the next section and the proof of Theorem 3, one can see that *most* of the restrictions performed are in fact setting variables to 0. Furthermore, by inspecting the flat used for the attack performed in [19] (section 5.3), one can see that it is of this form as well. Determining whether a given function represented by its ANF admits a 0-restriction  $f'$  on  $n - k$  variables with  $f'$  constant corresponds exactly to the hitting set problem, and this is well known to be **NP** complete [12]. Furthermore it remains **NP** complete even when restricted to quadratic functions (corresponding to the vertex cover problem). This stands in contrast to Proposition 1; for quadratic functions and general flats (as opposed to just 0-restrictions) the problem is polynomial time solvable. To the best of our knowledge, the computational complexity of the following problem is open (see also [10]): Given a function, represented by its ANF, find a large(st) flat on which the function is constant.

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<sup>3</sup> The constant 6.12 does not appear explicitly in these articles, however it can be derived using similar calculations as in the cited papers. This also follows from Theorem 6 later in this paper. We remark that 6.12 is not optimal.

### 3 Majority Has High Algebraic Thickness

For many functions, it is trivial to see that the ANF contains many terms, e.g. the function  $f(\mathbf{x}) = (1 + x_1)(1 + x_2) \cdots (1 + x_n)$ , which is 1 if and only if all the inputs are 0, contains all the possible  $2^n$  terms in its ANF. However, we are not aware of any explicit function along with a proof of a strong (e.g. exponential) lower bound on the algebraic thickness. Using a result from circuit complexity [20], it is straightforward to show that the *majority function*,  $MAJ_n$  has exponential algebraic thickness.  $MAJ_n$  is 1 if and only if at least half of the  $n$  inputs are 1. In the following, an  $AC_0[\oplus]$  circuit of depth  $d$  is a circuit with inputs  $x_1, x_2, \dots, x_n, (1 \oplus x_1), (1 \oplus x_2), \dots, (1 \oplus x_n)$ . The circuit contains  $\wedge, \vee, \oplus$  (AND, OR, XOR) gates of unbounded fan-in, and every directed path contains at most  $d$  edges. First we need the following simple proposition:

**Proposition 2.** *Let  $f: \mathbb{F}_2^n \rightarrow \mathbb{F}_2$  have  $\mathcal{T}(f) \leq T$ . Then  $f$  can be computed by an  $AC_0[\oplus]$  circuit of depth 3 with at most  $n + T + 1$  gates.*

*Proof.* Suppose  $f = g \circ A$  for some affine bijective mapping  $A$ . In the first layer (the layer closest to the inputs) one can compute  $A$  using  $n$  XOR gates of fan-in at most  $n$ . Then by computing all the monomials independently,  $g$  can be computed by an  $AC_0[\oplus]$  circuit of depth 2 using  $T$  AND gates with fan-in at most  $n$  and 1 XOR gate of fan-in  $T$ .  $\square$

Now we recall a result due to Razborov [20], see also [14, 12.24]

**Theorem 4 (Razborov).** *Every unbounded fan-in depth- $d$  circuit over  $\{\wedge, \vee, \oplus\}$  computing  $MAJ_n$  requires  $2^{\Omega(n^{1/(2d)})}$  gates.*

Combining these two results, we immediately have the following result that the majority function  $MAJ_n$  has high algebraic thickness.

**Proposition 3.**  $\mathcal{T}(MAJ_n) \geq 2^{\Omega(n^{1/6})}$ .

### 4 Algebraic Thickness and Normality

This section is devoted to showing that functions with algebraic thickness at most  $n^{3-\epsilon}$  are constant on flats of somewhat large dimensions. Furthermore our proof reveals a polynomial time algorithm to find such a subspace. In the following, a term of degree at least 3 will be called a *crucial* term, and for a function  $f$ , the number of crucial terms will be denoted  $\mathcal{T}^{\geq 3}(f)$ .

Our approach can be divided into two steps: First it uses 0-restrictions to obtain a quadratic function, and after this we can use Proposition 1. As implied by the relation between 0-restrictions and the hitting set problem, finding the optimal 0-restrictions is indeed a computationally hard task. Nevertheless, as we shall show in this section, the following greedy algorithm gives reasonable guarantees.

The greedy algorithm works by continually finding the variable that is contained in most crucial terms, and sets this variable to 0. It finishes when there are no crucial terms. We show that when the greedy algorithm finishes, the number of variables left,  $n'$ , is relatively large as a function of  $n$  (for a more precise statement, see Theorem 5). Notice that we are only interested in the behavior of  $n'$  as a function of  $n$ , and that this is not necessarily related to the approximation ratio of the greedy algorithm, which is known to be  $\Theta(\log n)$  [13].

We begin with a simple proposition about the greedy algorithm that will be useful throughout the section, and it gives a tight bound.

**Proposition 4.** *Let  $g: \mathbb{F}_2^n \rightarrow \mathbb{F}_2$  have  $\mathcal{T}^{\geq 3}(g) \geq m$ . Then some variable  $x_j$  is contained in at least  $\lceil 3\frac{m}{n} \rceil$  crucial terms.*

*Proof.* We can assume that no variable occurs twice in the same term. Hence the total number of variable occurrences in crucial terms is at least  $3m$ . By the pigeon hole principle, some variable is contained in at least  $\lceil 3\frac{m}{n} \rceil$  terms.  $\square$

The following lemma is a special case where a tight result can be obtained. The result applies to functions with relatively small thickness, and a later lemma reduces functions with somewhat larger thickness to this case.

**Lemma 1.** *Let  $c \leq \frac{2}{3}$  and let  $f: \mathbb{F}_2^n \rightarrow \mathbb{F}_2$  have  $\mathcal{T}^{\geq 3}(f) \leq cn$ . Then  $f$  has a 0-restriction  $f'$  on  $n' = n - \lceil \frac{3c-1}{5}n \rceil$  variables with  $\mathcal{T}^{\geq 3}(f') \leq \frac{n'}{3}$ .*

*Proof.* Let the greedy algorithm run until a function  $f'$  on  $n'$  variables with  $\mathcal{T}^{\geq 3}(f') \leq \frac{n'}{3}$  is obtained. By Proposition 4 we eliminate at least 2 terms in each step. The number of algorithm iterations is at most  $\lceil \frac{3c-1}{5}n \rceil$ . Indeed, let  $\lceil \frac{3c-1}{5}n \rceil = \frac{3c-1}{5}n + \delta$  for some  $0 \leq \delta < 1$ . After this number of iterations the number of variables left is  $n' = n - \frac{3c-1}{5}n - \delta = \frac{6-3c}{5}n - \delta$  and the number of critical terms is at most  $cn - 2\left(\frac{3c-1}{5}n - \delta\right) = \frac{2-c}{5}n - 2\delta$ . In particular  $\frac{n'}{3} \geq \frac{2-c}{5}n - 2\delta$ .  $\square$

Lemma 1 is essentially tight.

**Proposition 5.** *Let  $\frac{1}{3} < c \leq \frac{2}{3}$  be arbitrary but rational. Then for infinitely many values of  $n$ , there exists a function on  $n$  variables with  $\mathcal{T}^{\geq 3}(f) = cn$  such that every 0-restriction  $f'$  on  $n' > n - \lceil \frac{3c-1}{5}n \rceil$  variables has  $\mathcal{T}^{\geq 3}(f') > \frac{n'}{3}$ .*

*Proof.* Let  $\frac{1}{3} < c \leq \frac{2}{3}$  be fixed and consider the function on 6 variables:  $f(x) = x_1x_2x_3 + x_1x_4x_5 + x_2x_4x_6 + x_3x_5x_6$ . The greedy algorithm sets this functions to 0 by killing two variables, and this is optimal. Furthermore setting any one variable to 0 kills exactly two terms. Now consider the following function defined on  $n = 30m$  variables and having  $20m$  terms. For convenience we index the variables by  $x_{i,j}$  for  $1 \leq i \leq 5m$ ,  $1 \leq j \leq 6$ . Let  $g(x) = \sum_{i=1}^{5m} f(x_{i,1}, x_{i,2}, x_{i,3}, x_{i,4}, x_{i,5}, x_{i,6})$ . Again here the greedy algorithm is optimal, and setting  $6m$  variables to zero leaves  $n' = 24m$  variables and  $8m$  terms remaining. Thus, the bound from Lemma 1 is met with equality for  $c = \frac{2}{3}$ .

To see that it is tight for  $c < \frac{2}{3}$ , consider the function,  $\tilde{f}$  on  $n$  variables, where  $n$  is a multiple of 30 such that  $c\frac{4}{3}\frac{n}{2-c}$  is an integer. Run the greedy algorithm until the number of variables is  $\tilde{n}$  and  $\mathcal{T}^{\geq 3}(\tilde{f}) = c\tilde{n}$  (assuming  $c\tilde{n}$  is an integer). At this point  $\tilde{n} = \frac{4}{3}\frac{n}{2-c}$  and the number of terms left is  $c\tilde{n}$ . Again, by the structure of the function, setting any number,  $t$ , of the variables to 0 results in a function with  $\tilde{n} - t$  variables and at least  $c\tilde{n} - 2t$  terms. When  $t < \frac{(3c-1)\tilde{n}}{5}$ , we have  $c\tilde{n} - 2t > \frac{\tilde{n}-t}{3}$ . □

An immediate corollary to Lemma 1 is the following.

**Corollary 1.** *Let  $f: \mathbb{F}_2^n \rightarrow \mathbb{F}_2$  have  $\mathcal{T}^{\geq 3}(f) \leq \frac{2}{3}n$ . Then it is constant on a flat of dimension  $n' \geq \left\lfloor \frac{\frac{2}{3}\lfloor \frac{4}{5}n \rfloor}{2} \right\rfloor \geq \frac{4}{15}n - 2$ . Furthermore, such a flat can be found efficiently.*

*Proof.* First apply Lemma 1 to obtain a function on  $n' = \lfloor \frac{4}{5}n \rfloor$  variables with at most  $\frac{n'}{3}$  crucial terms. Now set one variable in each crucial term to 0, so after this we have at least  $\frac{2}{3}\lfloor \frac{4}{5}n \rfloor$  variables left and the remaining function is quadratic. Applying Theorem 1 gives the result. □

The following lemma generalizes the lemma above to the case with more terms. The analysis of the greedy algorithm uses ideas similar to those used in certain formula lower bound proofs, see e.g. [23] or [14, Section 6.3].



**Lemma 2.** *Let  $f: \mathbb{F}_2^n \rightarrow \mathbb{F}_2$  with  $\mathcal{T}^{\geq 3}(f) \leq n^{3-\epsilon}$ , for  $0 < \epsilon < 2$ . Then there exists a 0-restriction  $f'$  on  $n' = \lfloor \sqrt{\frac{2}{3}n^\epsilon} \rfloor$  variables with  $\mathcal{T}^{\geq 3}(f') \leq \frac{2}{3}n'$ .*

*Proof.* Let  $\mathcal{T}^{\geq 3}(f) = T$ . Then, by Proposition 4. Setting the variable contained in the largest number of terms to 0, the number of crucial terms left is at most

$$T - \frac{3T}{n} = T \cdot \left(1 - \frac{3}{n}\right) \leq T \cdot \left(\frac{n-1}{n}\right)^3.$$

Applying this inequality  $n - n'$  times yields that after  $n - n'$  iterations the number of crucial terms left is at most

$$T \cdot \left(\frac{n-1}{n}\right)^3 \left(\frac{n-2}{n-1}\right)^3 \cdots \left(\frac{n'}{n'+1}\right)^3 = T \cdot \left(\frac{n'}{n}\right)^3.$$

When  $n' = \sqrt{\frac{2}{3}n^\epsilon}$  and  $T = n^{3-\epsilon}$ , this is at most  $\frac{2}{3}n'$ .  $\square$

*Remark:* A previous version of this paper [2], contained a version of the lemma with a proof substantially more complicated. We thank anonymous reviewers for suggesting this simpler proof.

It should be noted that Lemma 2 cannot be improved to the case where  $\epsilon = 0$ , no matter what algorithm is used to choose the 0-restriction. To see this consider the function containing all degree three terms. For this function, *any* 0-restriction leaving  $n'$  variables will have at least  $\binom{n'}{3}$  crucial terms. This suggests that for handling functions with larger algebraic thickness, one should use restrictions other than just 0-restrictions.

Combining the lemma with Corollary 1, we get the following theorem.

**Theorem 5.** *Let  $\mathcal{T}(f) = n^{3-\epsilon}$  for  $0 < \epsilon < 2$ . Then there exists a flat of dimension at least  $\frac{4}{15}\sqrt{\frac{2}{3}n^\epsilon} - 3$ , such that when restricted to this flat,  $f$  is constant. Furthermore this flat can be found efficiently.*

This improves on Theorem 3 for functions with algebraic thickness  $n^s$  for  $1 \leq s \leq 2.83$ , and the smaller  $s$ , the bigger the improvement, e.g. for  $\mathcal{T}(f) \leq n^2$ , our bound guarantees  $\mathcal{N}(f) \in \Omega(n^{1/2})$ , compared to  $\Omega(n^{1/8})$ .

#### 4.1 Normal Functions with low sparsity

The purpose of this section is first to show that the result from Theorem 5 is at most a factor of  $\Theta(\sqrt{n})$  from being tight. More precisely, we show

that for any  $2 < s \leq 3$  there exist functions with thickness at most  $n^s$  that are nonconstant on flats of dimension  $O(n^{2-\frac{s}{2}})$ . Notice that this contains Theorem 2 as a special case where  $s = 3$ .

**Theorem 6.** *For any  $2 < s \leq 3$ , for sufficiently large  $n$ , there exist functions with degree 3 and algebraic thickness at most  $n^s$  that, for sufficiently large  $n$ , remain nonconstant on all flats of dimension  $6.12n^{2-\frac{s}{2}}$ .*

*Proof.* Using the probabilistic method, we endow the set of all Boolean functions of degree 3 with a probability distribution  $\mathcal{D}$ , and show that a function under  $\mathcal{D}$  has the promised normality with high probability.

First we describe  $\mathcal{D}$ . Then, we fix an arbitrary  $k$ -dimensional flat  $E$ , and bound the probability that a random  $f$  chosen according to  $\mathcal{D}$  is constant on  $E$ . We show that for  $k = Cn^{2-s/2}$ , where the constant  $C$  is determined later, this probability is sufficiently small that a union bound over all possible choices of  $E$  gives the desired result.

We define  $\mathcal{D}$  by describing the probability distribution on the ANF. We let each possible degree 3 term be included with probability  $\frac{1}{2n^{3-s}}$ . The expected number of terms is thus  $\frac{1}{2}n^{s-3}\binom{n}{3} \leq n^s/12$ , and the probability of having more than  $n^s$  terms is less than 0.001 for large  $n$ . Now let  $E$  be an arbitrary but fixed  $k$ -dimensional flat.

One way to think of a function restricted to a  $k$ -dimensional flat is that it can be obtained by a sequence of  $n - k$  affine variable substitutions of the form  $x_i := \sum_{j \in S} x_j + c$ . This changes the ANF of the function since  $x_i$  is no longer a “free” variable. Assume without loss of generality that we substitute for the variables  $x_n, \dots, x_{k+1}$  in that order. Initially we start with the function  $f$  given by

$$f(x) = \sum_{\{a,b,c\} \subseteq [n]} I_{abc} x_a x_b x_c,$$

where  $I_{abc}$  is the indicator random variable, indicating whether the  $x_a x_b x_c$  is contained in the ANF. Suppose we perform the  $n - k$  restrictions and obtain the function  $\tilde{f}$ . The ANF of  $\tilde{f}$  is given by

$$f(x) = \sum_{\{a,b,c\} \subseteq [k]} \left( I_{abc} + \sum_{s \in S_{abc}} I_s \right) x_a x_b x_c,$$

where  $S_{abc}$  is some set of indicator random variables depending on the restrictions performed. It is important that  $I_{abc}$ , the indicator random variable corresponding to  $x_a x_b x_c$ , for  $\{a, b, c\} \subseteq [k]$  is *only* occurring at

$x_ax_bx_c$ . Hence we conclude that independently of the outcome of all the indicator random variables  $I_{a'b'c'}$  with  $\{a', b', c'\} \not\subseteq [k]$ , we have that the marginal probability for any  $I_{abc}$  with  $\{a, b, c\} \subseteq [k]$  occurring remains at least  $\frac{1}{2n^{3-s}}$ .

Define  $t = \binom{k}{3}$  random variables,  $Z_1, \dots, Z_t$ , one for each potential term in the ANF of  $\tilde{f}$ , such that  $Z_j = 1$  if and only if the corresponding term is present in the ANF, and 0 otherwise. The obtained function is only constant if there are no degree 3 terms, so the probability of  $\tilde{f}$  being constant is thus at most

$$\mathbb{P}[Z_1 = \dots = Z_t = 0] \leq \left(1 - \frac{1}{2n^{3-s}}\right)^t \leq \exp\left(-\frac{C^3}{54}n^{3-s/2}\right).$$

The number of choices for  $E$  is at most  $2^{n(k+1)}$ , so the probability that  $f$  becomes constant on *some* affine flat of dimension  $k$  is at most

$$\exp\left(-\frac{C^3}{54}n^{3-s/2} + C \ln(2)n^{3-s/2} + n\right).$$

Now if  $C > \sqrt{54 \ln(2)} \approx 6.11..$ , this quantity tends to 0. We conclude that with high probability the function obtained has algebraic thickness at most  $n^s$  and normality at most  $6.12n^{2-\frac{s}{2}}$ .  $\square$

There is factor of  $\Theta(\sqrt{n})$  between the existence guaranteed by Theorem 5 and Theorem 6 and we leave it as an interesting problem to close this gap.

The algorithm studied in this paper works by setting variables to 0 until all remaining terms have degree at most 2, and after that appealing to Theorem 1. A proof similar to the previous shows that among such algorithms, the bound from Theorem 5 is very close to being asymptotically tight. The proof can be found in [2].

**Theorem 7.** *For any  $2 < s < 3$ , there exist functions with degree 3 and algebraic thickness at most  $n^s$  that have degree 3 on any 0-restriction of dimension  $3\sqrt{\ln nn}^{\frac{3-s}{2}}$ .*

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