

MULTIPLICATIVE COMPLEXITY OF VECTOR VALUED BOOLEAN FUNCTIONS

MAGNUS GAUSDAL FIND AND JOAN BOYAR

July 24, 2015

Abstract. We consider the multiplicative complexity of Boolean functions with multiple bits of output, studying how large a multiplicative complexity is necessary and sufficient to provide a desired nonlinearity. For so-called $\Sigma\Pi\Sigma$ circuits, we show that there is a tight connection between error correcting codes and circuits computing functions with high nonlinearity. Combining this with known coding theory results, we show that functions with n inputs and n outputs with the highest possible nonlinearity must have at least $2.32n$ AND gates. We further show that one cannot prove stronger lower bounds by only appealing to the nonlinearity of a function; we show a bilinear circuit computing a function with almost optimal nonlinearity with the number of AND gates being exactly the length of such a shortest code.

Additionally we provide a function which, for general circuits, has multiplicative complexity at least $2n - 3$.

Finally we study the multiplicative complexity of “almost all” functions. We show that every function with n bits of input and m bits of output can be computed using at most $2.5(1 + o(1))\sqrt{m2^n}$ AND gates.

1. Definitions and Preliminaries

Let \mathbb{F}_2 be the finite field of order 2 and \mathbb{F}_2^n the n -dimensional vector space over \mathbb{F}_2 .

We denote by $[n]$ the set $\{1, \dots, n\}$. An (n, m) -function is a mapping from \mathbb{F}_2^n to \mathbb{F}_2^m and we refer to these as the *Boolean functions*. When $m > 1$ we say that the function is *vector valued*.

It is well known that every $(n, 1)$ -function f can be written uniquely as

a multilinear polynomial over \mathbb{F}_2

$$f(x_1, \dots, x_n) = \sum_{X \subseteq [n]} \alpha_X \prod_{i \in X} x_i.$$

This polynomial is called the *Zhegalkin polynomial* or the *algebraic normal form* of f . For the rest of this paper most, but not all, arithmetic will be in \mathbb{F}_2 . We trust that the reader will find it clear whether arithmetic is in \mathbb{F}_2 , \mathbb{F}_{2^n} , or \mathbb{R} when not explicitly stated, and will not address it further.

The *degree* of an $(n, 1)$ -function f is the largest $|X|$ such that $\alpha_X = 1$. For an (n, m) -function f , we let f_i be the $(n, 1)$ -function defined by the i th output bit of f , and say that the degree of f is the largest degree of f_i for $i \in [m]$. A function is *affine* if it has degree 1, and *quadratic* if it has degree 2. For $T \subseteq [m]$ we let

$$f_T = \sum_{i \in T} f_i,$$

and for $\mathbf{v} \in \mathbb{F}_2^n$ we let $|\mathbf{v}|$ denote the *Hamming weight* of \mathbf{v} , that is, the number of nonzero entries in \mathbf{v} , and let $|\mathbf{u} + \mathbf{v}|$ be the *Hamming distance* between the two vectors \mathbf{u} and \mathbf{v} .

Nonlinearity of Boolean Functions We will use several facts on the nonlinearity of Boolean functions. We refer to the two chapters in Carlet (2010a,b) for proofs and references.

The *nonlinearity* of an $(n, 1)$ -function f is the Hamming distance to the closest affine function, more precisely

$$NL(f) = 2^n - \max_{\mathbf{a} \in \mathbb{F}_2^n, b \in \mathbb{F}_2} |\{\mathbf{x} \in \mathbb{F}_2^n \mid \langle \mathbf{a}, \mathbf{x} \rangle + b = f(\mathbf{x})\}|,$$

where $\langle \mathbf{a}, \mathbf{x} \rangle = \sum_{i=1}^n a_i x_i$. For an (n, m) -function f , the nonlinearity is defined as

$$NL(f) = \min_{T \subseteq [m], T \neq \emptyset} \{NL(f_T)\}.$$

The nonlinearity of an (n, m) -function is always between 0 and $2^{n-1} - 2^{\frac{n}{2}-1}$. The (n, m) -functions meeting this bound are called *bent functions*. Bent $(n, 1)$ functions exist if and only if n is even. A standard example of a bent $(n, 1)$ -function is the *inner product*, on $n = 2k$ variables, defined as:

$$IP_{2k}(x_1, \dots, x_k, y_1, \dots, y_k) = \langle \mathbf{x}, \mathbf{y} \rangle.$$

This function is clearly quadratic. If we identify \mathbb{F}_2^n with \mathbb{F}_{2^n} , a standard example of a bent $(2n, n)$ -function is the *finite field multiplication* function:

$$(1.1) \quad f(\mathbf{x}, \mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$$

where multiplication is in \mathbb{F}_{2^n} .

If $n = m$, $NL(f)$ is at most $2^{n-1} - 2^{\frac{n-1}{2}}$, see Chabaud & Vaudenay (1995). Functions meeting this bound are called *almost bent*. These exist only for odd n . As remarked by Carlet, this name is a bit misleading since the name indicates that they are suboptimal, which they are not. Again, if we identify \mathbb{F}_2^n and \mathbb{F}_{2^n} , for $1 \leq i \leq \frac{n-1}{2}$ and $\gcd(i, n) = 1$, the so called *Gold functions* defined as

$$(1.2) \quad G(\mathbf{x}) = \mathbf{x}^{2^i+1} = \mathbf{x} \cdot (\mathbf{x}^{2^i})$$

are almost bent. This function is quadratic since the \mathbb{F}_{2^n} -operator $\mathbf{x} \mapsto \mathbf{x}^2$ is affine in when considered as an operator on \mathbb{F}_2^n , and each output bit of finite field multiplication is quadratic in the inputs, see also Carlet (2010b).

Multiplicative Complexity and Circuit Classes In this paper we consider multiple classes of circuits:

- An *XOR-AND circuit* is a Boolean circuit where each of the gates is either \oplus (XOR, addition in \mathbb{F}_2), \wedge (AND, multiplication in \mathbb{F}_2) or the constant 1. The XOR gates may have unbounded fanin, and the AND gates have fanin 2;
- a $\Sigma\Pi\Sigma$ circuit is a circuit containing three layers of gates. The first layer contains XOR gates, the second contains AND gates and the third contains XOR gates. All gates are allowed unbounded fanin;
- a circuit is *quadratic* if it is both a $\Sigma\Pi\Sigma$ circuit and an XOR-AND circuit. This is equivalent to saying that all gates in the circuit compute functions of degree at most 2;
- a quadratic circuit is *bilinear* if the input is partitioned into two sets, and each input to an AND gate is a linear combination of variables from one of these two sets, with the other input using the opposite set of the partition.

The *multiplicative complexity* of an (n, m) -function with respect to a circuit class \mathcal{C} is the smallest number of AND gates in any circuit from \mathcal{C} computing f . If the circuit class \mathcal{C} is not specified, we refer to the multiplicative complexity with respect to XOR-AND circuits.

We notice the hierarchy between the circuits: any bilinear circuit is a quadratic circuit and any quadratic circuit is both an $\Sigma\Pi\Sigma$ and an XOR-AND circuit.

Some functions have higher multiplicative complexity with respect to XOR-AND circuits than to $\Sigma\Pi\Sigma$ circuits. Examples of this include the function $f(\mathbf{x}) = x_1x_2\cdots x_n$, having multiplicative complexity, $n - 1$ and 1 , respectively. On the other hand, some functions have much lower multiplicative complexity with respect to XOR-AND circuits than to $\Sigma\Pi\Sigma$ circuits. An example of this is the *majority* function, known to have multiplicative complexity close to n (Boyar & Peralta 2008) with respect to XOR-AND circuits, but exponentially many AND gates are needed in any $\Sigma\Pi\Sigma$ circuit (see Razborov (1987) and Jukna (2012)). Note that the exponential lower bound on the total number of gates implies an exponential lower bound on the number of AND gates. To see this, note that the output level contains only one gate, and if there were more than n inputs to any AND gate, they would be linearly dependent. The AND of linearly dependent linear functions can either be expressed with fewer inputs or has an output of zero and can be removed.

Relationship Between Nonlinearity and Multiplicative Complexity

Some relations between nonlinearity and multiplicative complexity are known (see also Carlet (2002)).

PROPOSITION 1.3 (Zheng *et al.* 1999). *Let f be an $(n, 1)$ -function. Suppose there exists an affine subspace U of dimension k such that f is affine on the points in U . Then f has nonlinearity at most $2^{n-1} - 2^{k-1}$.*

If a function has multiplicative complexity $n - k$, then there exists an affine subspace of dimension k on which f is affine (see e.g. Boyar & Peralta (2008)). Applying this the following proposition is immediate.

PROPOSITION 1.4. *Let f be an $(n, 1)$ -function with multiplicative complexity M . Then the $NL(f) \leq 2^{n-1} - 2^{n-M-1}$.*

We remark that this result holds for both $\Sigma\Pi\Sigma$ circuits and XOR-AND circuits. It is a slight generalization of a result given in Boyar *et al.* (2013) where we showed a similar relation. The proof of the latter result holds however only for XOR-AND circuits. There it is also shown that the above results are tight; for every $M \leq \frac{n}{2}$ there exists an explicit function meeting the bound with equality, for both $\Sigma\Pi\Sigma$ and XOR-AND circuits.

1.1. Linear Codes. Most bounds in this paper will come from coding theory. In this subsection, we briefly review the necessary facts. For more information, see chapter 17 in Jukna (2011) or the older but comprehensive (Sloane & MacWilliams 1977).

A *linear (error correcting) code* of length s is a linear subspace, \mathcal{C} of \mathbb{F}_2^s . The *dimension* of a code is the dimension of the subspace, \mathcal{C} , and the elements of \mathcal{C} are called *codewords*. The (minimum) *distance* d of \mathcal{C} is defined as

$$d = \min_{\mathbf{x} \neq \mathbf{y} \in \mathcal{C}} |\mathbf{x} + \mathbf{y}|.$$

The following fact is well known

PROPOSITION 1.5. *For every linear code, \mathcal{C} , the distance is exactly the minimum weight among non-zero codewords.*

Let $L(m, d)$ be the length of the shortest linear m -dimensional code over \mathbb{F}_2 with distance d . We will use lower and upper bounds on $L(m, d)$. One lower bound is the following (McEliece *et al.* 1977), see also Sloane & MacWilliams (1977), page 563.

THEOREM 1.6 (McEliece, Rodemich, Rumsey, Welch). *For $0 < \delta < 1/2$, let $\mathcal{C} \subseteq \{0, 1\}^s$ be a linear code with dimension m and distance δs . Then the rate $R = \frac{m}{s}$ of the code satisfies $R \leq \min_{0 \leq u \leq 1-2\delta} B(u, \delta)$, where $B(u, \delta) = 1 + h(u^2) - h(u^2 + 2\delta u + 2\delta)$, $h(x) = H_2\left(\frac{1-\sqrt{1-x}}{2}\right)$, and $H_2(x) = -x \log x - (1-x) \log(1-x)$.*

An upper bound is the following, see Jukna (2011).

THEOREM 1.7 (Gilbert-Varshamov). *A linear code $\mathcal{C} \subseteq \{0, 1\}^s$ of dimension m and distance d exists provided that $\sum_{i=0}^{d-2} \binom{s-1}{i} < 2^{s-m}$.*

2. Introduction

Cryptographic functions such as encryption functions should have high non-linearity to be resistant against linear and differential attacks (see again Carlet (2010b) and the references therein). This is an explicit design criteria for modern cryptographic systems, such as AES, (Daemen & Rijmen 2002), which has been used as a benchmark for several implementations of homomorphic encryption.

In several practical settings, such as homomorphic encryption and secure multiparty computation (see e.g. Vaikuntanathan (2011) and Kolesnikov & Schneider (2008)), the number of AND gates is significantly more important than the number of XOR gates, hence one is interested in (n, m) -functions with as few AND gates as possible. For many such protocols, it is not just the *number* of AND gates that matters, but also the *AND depth*. For example, in several protocols for secure multiparty computation the number of AND gates

is proportional to the number of bits sent, and the AND depth corresponds to the number of rounds in the protocol (see e.g. Lepoint & Paillier (2013)), and in typical protocols for homomorphic encryption, the norm of the noise after an AND gate is the product of the norms of the noise from the inputs, so the AND depth greatly affects the number of expensive bootstrappings, relinearizations, and/or modulus reductions which are necessary (see e.g. Doröz *et al.* (2014)). For more examples we refer to (Albrecht *et al.* 2015) and the references therein.

A natural question to ask is how nonlinearity and multiplicative complexity are related to each other: how large does one measure need to be in order for the other to have at least a certain value? As stated in Section 1, for every desired nonlinearity, it is known exactly how many AND gates are necessary and sufficient for an $(n, 1)$ -function to achieve this. We study this same question for functions with multiple bits of output.

Our Contributions Let f be an (n, m) -function with nonlinearity $2^{n-1} - 2^{n-M-1}$. We show that any $\Sigma\Pi\Sigma$ circuit with s AND gates computing f defines an m -dimensional linear code in \mathbb{F}_2^s with distance M , so lower bounds on the size of such codes show lower bounds on the number of AND gates in such a circuit. This means that a proof of high nonlinearity for a function is automatically a lower bound on the multiplicative complexity for this class of circuits. We instantiate this result for the specific case of quadratic circuits. First, the so called *Gold* functions with n bits of input and n bits of output are quadratic and almost bent. Using known coding theory bounds, we conclude that any quadratic circuit computing such functions must have at least $2.32n$ AND gates. To the best of our knowledge this is the best lower bound for such circuits. Similarly a well known lower bound for finite field multiplication (Brown & Dobkin 1980; Lempel *et al.* 1983) follows as a direct corollary.

On the other hand, we show that appealing only to the nonlinearity of a function cannot lead to much stronger lower bounds on the multiplicative complexity by showing the existence of *quadratic* (in fact, *bilinear*) *circuits* from n bits to n bits with nonlinearity at least $2^{n-1} - 2^{\frac{n}{2}+3\sqrt{n}}$ containing s AND gates where s is the length of a shortest n -dimensional code of distance $\frac{n}{2}$. The Gilbert-Varshamov bound gives that $s \leq 2.95n$.

Although almost all Boolean functions with n inputs and one output have multiplicative complexity at least $2^{n/2} - O(n)$ (Boyar *et al.* 2000), no concrete function of this type has been shown to have multiplicative complexity more than $n - 1$. We give a concrete function with n inputs and n outputs with multiplicative complexity at least $2n - 3$. To the best of our knowledge this

is the best lower bound for the multiplicative complexity for an explicit (n, n) -function.

Finally we study the worst case multiplicative complexity of vector valued functions. We show that almost every (n, m) -function has multiplicative complexity at least $(1 - o(1))\sqrt{m2^n}$ and that *every* such function can be computed using an XOR-AND circuit with at most $(2.5 + o(1))\sqrt{m2^n}$ AND gates.

2.1. Related Results. Previous results showing relations between error correcting codes and bilinear and quadratic circuits include the work of Brown & Dobkin (1980); Lempel *et al.* (1983) where it is shown that a bilinear or quadratic circuit computing finite field multiplication of two \mathbb{F}_{q^n} elements induces an error correcting code over \mathbb{F}_q of dimension n and distance n . For $q = 2$, Theorem 1.6 implies that such a circuit must have at least $3.52n$ multiplications (AND gates). If n is the number of input bits, this corresponds to a lower bound of $1.76n$. For $q > 2$, the gates (or lines in a straight-line program) have field elements as inputs, and the total number of multiplications and divisions is counted. A lower bound of $3n - o(n)$ for bilinear circuits was shown in Kaminski & Bshouty (1989) and extend it to general circuits (Bshouty & Kaminski 2006). This proof is not based on coding theoretic techniques, but rather the study of Hankel matrices related to the bilinear transformation.

A different relation between general Boolean (not just quadratic) $(n, 1)$ -functions and error codes comes from the following observation: Suppose some (n, m) -function f has a certain nonlinearity D . If we identify the functions $f_1, \dots, f_m, x_1, \dots, x_n$ and the constant 1 with their truth tables as vectors in $\mathbb{F}_2^{2^n}$, then $\mathcal{C} = \text{span}\{f_1, \dots, f_m, x_1, \dots, x_n, 1\}$ is a code in $\mathbb{F}_2^{2^n}$ with dimension $n + m + 1$ and distance D , and limitations and possibilities for codes transfer to results on nonlinearity (see the survey (Carlet 2010b) and the references therein). However this says nothing about the multiplicative complexity of the function f .

The structure of quadratic circuits has itself been studied by Mirwald and Schnorr (Mirwald & Schnorr 1992). Among other things they show that for quadratic $(n, 1)$ - and $(n, 2)$ -functions, quadratic circuits are optimal. It is still not known whether this is true for (n, m) -functions in general.

3. $\Sigma\Pi\Sigma$ Circuits: Multiplicative and Nonlinearity

This section is devoted to showing a relation between the nonlinearity and the multiplicative complexity of $\Sigma\Pi\Sigma$ circuits. We first show a connection between nonlinearity, multiplicative complexity and certain linear codes. Ap-

plying this connection, Theorem 1.6 gives a relationship between nonlinearity and multiplicative complexity for any quadratic (n, m) -function.

THEOREM 3.1. *Let the (n, m) -function, f , have $NL(f) \geq 2^{n-1} - 2^{n-M-1}$, where $M \leq \frac{n}{2}$. Then a $\Sigma\Pi\Sigma$ circuit with s AND gates computing f exhibits an m -dimensional linear code over \mathbb{F}_2^s with distance M .*

PROOF. Let C be a $\Sigma\Pi\Sigma$ circuit with s AND gates computing f , and let A_1, \dots, A_s , be the AND gates. Since C is $\Sigma\Pi\Sigma$, for each $i \in [m]$ there exist $S_i \subseteq [s]$ and $X_i \subseteq [n]$ such that f_i can be written as

$$f_i = \sum_{j \in S_i} A_j + \sum_{j \in X_i} x_j.$$

Without loss of generality, we can assume that $X_i = \emptyset$ for all i , since both nonlinearity and multiplicative complexity are invariant under the addition of affine terms. For each $i \in [m]$, we define the vector $\mathbf{v}_i \in \mathbb{F}_2^s$, where $\mathbf{v}_{i,j} = 1$ if and only if there is a directed path from A_j to the i th output. By the nonlinearity of f , we have that for each $i \in [m]$,

$$NL(f_i) \geq 2^{n-1} - 2^{n-M-1}.$$

Applying Proposition 1.4, the multiplicative complexity of f_i is at least M , hence $|\mathbf{v}_i| \geq M$. Similarly, for any nonempty $T \subseteq [m]$ we can associate a vector \mathbf{v}_T by setting

$$\mathbf{v}_T = \sum_{i \in T} \mathbf{v}_i.$$

Since the circuit is $\Sigma\Pi\Sigma$, it holds that if $|\mathbf{v}_T| \leq p$, the multiplicative complexity of $f_T = \sum_{i \in T} f_i$ is at most p . Applying the definition of nonlinearity to f_T , $NL(f_T) \geq 2^{n-1} - 2^{n-M-1}$. Proposition 1.4 implies that the multiplicative complexity of f_T is at least M , so we have that $|\mathbf{v}_T| \geq M$ when $T \neq \emptyset$.

In conclusion, every nonzero vector in the m dimensional vector space $\mathcal{C} = \text{span}_{\mathbb{F}_2}\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ has Hamming weight at least M . By Proposition 1.5, \mathcal{C} is a linear code with dimension m and distance at least M . \square

Applying this theorem to quadratic almost bent functions, we have that a quadratic circuit computing such a function has at least $L(n, \frac{n-1}{2})$ AND gates. Combining this with Theorem 1.6, calculations, which we include for the sake of completeness, show:

COROLLARY 3.2. *Any quadratic circuit computing an almost bent (n, n) -function has at least $L(n, \frac{n-1}{2})$ AND gates. For sufficiently large n , $L(n, \frac{n-1}{2}) > 2.32n$.*

PROOF. Recall that almost bent functions have nonlinearity $2^{n-1} - 2^{\frac{n-1}{2}}$, so in terms of Theorem 3.1, we have $M = \frac{n-1}{2}$. Suppose for the sake of contradiction that $L(n, \frac{n-1}{2}) \leq 2.32n$ for infinitely many values of n . Then we have an n -dimensional code on $\mathbb{F}_2^{2.32n}$ with distance $\frac{n-1}{2}$. In Theorem 1.6, we have

$$\delta = \frac{1}{2 \cdot 2.32} \approx 0.2155.$$

The rate of the code is $R = \frac{1}{2.32} > 0.431$. Choosing $u = 0.32$ shows that the rate R satisfies

$$0.431 < R \leq B(0.32, 0.2155) < 0.42,$$

contradicting the assumption that $L(n, \frac{n-1}{2}) \leq 2.32n$. \square

The corollary above applies to e.g. the almost bent Gold functions G defined in Eqn. (1.2). For bent $(2n, n)$ -functions, using Theorem 3.1 with $M = n$ and applying Theorem 1.6, calculations, which are not new, but again included for the sake of completeness, show:

COROLLARY 3.3. *A quadratic circuit computing any bent $(2n, n)$ -function has at least $L(n, n)$ AND gates. For sufficiently large n , $L(n, n) > 3.52n$.*

PROOF. As in the proof above, in terms of Theorem 3.1, we have $m = n$ and $M = n$. Again suppose for the sake of contradiction that $L(n, n) \leq 3.52$ for infinitely many values of n . In Theorem 1.6, for sufficiently large n we have that

$$R = \delta = \frac{1}{3.52} \approx 0.28409.$$

Choosing $u = 0.4$ in Theorem 1.6 shows that the rate R satisfies

$$0.28409 < R \leq B(0.4, 0.28409) \approx 0.28260 < 0.284.$$

contradicting the assumption that $L(n, n) \leq 3.52$. \square

This applies to e.g. the finite field multiplication function as defined in Eqn. (1.1), reproving the known result on multiplicative complexity for quadratic circuits for field multiplication mentioned in Section 2.1.

For both Corollary 3.2 and Corollary 3.3, any improved lower bounds on codes lengths would give an improved lower bound on the multiplicative complexity. For Corollary 3.2 this technique cannot prove substantially better

lower bounds than $L(n, \frac{n-1}{2})$. Theorem 1.7 implies that $L(n, \frac{n-1}{2}) \leq 2.95n$. Below we show that this is not merely a limitation of the proof strategy; there exist quadratic circuits with $L(n, \frac{n-1}{2})$ AND gates with nonlinearity close to the optimal. To the best of our knowledge this is the first example of highly nonlinear (n, n) -functions with linear multiplicative complexity, and therefore it might be a useful building block for cryptographic purposes.

Before proving the next theorem, we need a technical lemma on the probability that a random matrix has small rank. A simple proof of this can be found in e.g. Komargodski *et al.* (2013).

LEMMA 3.4 (Komargodski, Raz, Tal). *A random $k \times k$ matrix has rank at most d with probability at most $2^{k-(k-d)^2}$.*

THEOREM 3.5. *There exist (n, n) -functions with multiplicative complexity at most $L(n, \frac{n-1}{2})$ and nonlinearity at least $2^{n-1} - 2^{\frac{n}{2}+3\sqrt{n}-1}$.*

PROOF. For simplicity we show the upper bound for $L(n, \frac{n}{2})$ AND gates. It is elementary to verify that it holds for $L(n, \frac{n-1}{2})$ AND gates as well. We give a probabilistic construction of a quadratic (in fact, bilinear) circuit with $s = L(n, \frac{n}{2})$ AND gates, then we show that with high probability, the function computed by this circuit has the desired nonlinearity.

For the construction of the circuit, we first define the value computed by the i th AND gate as $A_i(\mathbf{x}) = L_i(\mathbf{x})R_i(\mathbf{x})$ where L_i is a random sum over $x_1, \dots, x_{n/2}$ and R_i is a random sum over $x_{n/2+1}, \dots, x_n$. In the following, we will identify sums over $\mathbf{x}_1, \dots, \mathbf{x}_n$ with vectors in \mathbb{F}_2^n and sums over A_1, \dots, A_s with vectors in \mathbb{F}_2^s .

Let \mathcal{C} be an n -dimensional code of length $L(n, \frac{n}{2})$ with distance $\frac{n}{2}$ and let $\mathbf{y}_1, \dots, \mathbf{y}_n \in \mathbb{F}_2^s$ be a basis for \mathcal{C} . Now we define the corresponding sums over A_1, \dots, A_s to be the outputs computed by the circuit. This completes the construction of the circuit. Now fix $r(\mathbf{x}) \in \text{span}_{\mathbb{F}_2}\{\mathbf{y}_1, \dots, \mathbf{y}_n\}$, $r \neq \mathbf{0}$. We want to show that r has the desired nonlinearity with high probability. By an appropriate relabeling of the AND gates, we can write r as

$$(3.6) \quad r(\mathbf{x}) = \sum_{i=1}^q A_i(\mathbf{x}) = \sum_{i=1}^q L_i(\mathbf{x})R_i(\mathbf{x})$$

for some $q \geq \frac{n}{2}$. We now assume that

$$(3.7) \quad t = \text{rk}\{R_1, \dots, R_q\} \geq \frac{n}{2} - \frac{3\sqrt{n}}{2}.$$

At the end of the proof, we will show that this is true with high probability. Again by an appropriate relabeling, we let $\{R_1, \dots, R_t\}$ be a basis of

$\text{span}\{R_1, \dots, R_q\}$. If $q > t$, for $j > t$, we can write $R_j = \sum_{i=1}^t \alpha_{j,i} R_i$. In particular for $j = q$, we can substitute this into (3.6) and obtain

$$r(\mathbf{x}) = \sum_{i=1}^{q-1} (L_i(\mathbf{x}) + \alpha_{q,i} L_q(\mathbf{x})) R_i(\mathbf{x})$$

where we let $\alpha_{q,i} = 0$ for $i > t$. If $\{L_1, \dots, L_q\}$ are independently, uniformly randomly distributed, then so are $\{L_1 + \alpha_{q,1} L_q, \dots, L_{q-1} + \alpha_{q,q-1} L_q\}$. Continuing this process, we get that for $\frac{n}{2} \geq t \geq \frac{n}{2} - \frac{3\sqrt{n}}{2}$, there are sums $L'_1, \dots, L'_t, R'_1, \dots, R'_t$ such that

$$r(\mathbf{x}) = \sum_{i=1}^t L'_i(\mathbf{x}) R'_i(\mathbf{x})$$

where the $\{L'_1, \dots, L'_t\}$ are independently, uniformly random and the $\{R'_1, \dots, R'_t\}$ are linearly independent. We now further assume that

$$(3.8) \quad u = \text{rk}(L'_1, \dots, L'_t) \geq t - \frac{3\sqrt{n}}{2}.$$

Again, we will show at the end of this proof that this is true with high probability. Applying a similar procedure as above, we get that for some

$$u \geq t - \frac{3\sqrt{n}}{2} \geq \frac{n}{2} - 3\sqrt{n}$$

there exist sums $\tilde{L}_1, \dots, \tilde{L}_u$ and $\tilde{R}_1, \dots, \tilde{R}_u$, such that

$$r(\mathbf{x}) = \sum_{i=1}^u \tilde{L}_i(\mathbf{x}) \tilde{R}_i(\mathbf{x}),$$

where all $\tilde{L}_1, \dots, \tilde{L}_u$ and all $\tilde{R}_1, \dots, \tilde{R}_u$ are linearly independent. Thus, there exists a linear bijection $(\mathbf{x}_1, \dots, \mathbf{x}_n) \mapsto (\mathbf{z}_1, \dots, \mathbf{z}_n)$ with $\mathbf{z}_1 = \tilde{L}_1, \dots, \mathbf{z}_u = \tilde{L}_u, \mathbf{z}_{u+1} = \tilde{R}_1, \dots, \mathbf{z}_{2u} = \tilde{R}_u$, such that

$$\tilde{r}(\mathbf{z}) = \mathbf{z}_1 \mathbf{z}_{u+1} + \dots, \mathbf{z}_u \mathbf{z}_{2u}$$

where r and \tilde{r} are equivalent up to a linear bijection on the inputs. Since nonlinearity is invariant under linear bijections, we just need to determine the nonlinearity of \tilde{r} . Given the high nonlinearity of IP_n , it is elementary to verify that

$$NL(\tilde{r}) = 2^{n-2u} (2^{2u-1} - 2^{u-1}) = 2^{n-1} - 2^{n-u-1}.$$

If $u \geq \frac{n}{2} - 3\sqrt{n}$, this is at least $2^{n-1} - 2^{\frac{n}{2}+3\sqrt{n}-1}$.

Now it remains to show that the probability of either (3.7) or (3.8) occurring is so small that a union bound over all the $2^n - 1$ choices of r gives that with high probability, *every* $r \in \text{span}\{\mathbf{y}_1, \dots, \mathbf{y}_n\}$ has at least the desired nonlinearity.

For (3.7), we can think of the $q \geq \frac{n}{2}$ vectors R_1, \dots, R_q as rows in a $q \times \frac{n}{2}$ matrix. We will consider the upper left $\frac{n}{2} \times \frac{n}{2}$ submatrix. By Lemma 3.4 this has rank at most $\frac{n}{2} - \frac{3\sqrt{n}}{2}$ with probability at most

$$2^{\frac{n}{2} - \left(\frac{n}{2} - \left(\frac{n}{2} - \frac{3\sqrt{n}}{2}\right)\right)^2} = 2^{\frac{n}{2} - \frac{9n}{4}} = 2^{-\frac{7n}{4}}$$

Similarly for (3.8) we can consider the $\frac{n}{2} \geq t \geq \frac{n}{2} - \frac{3\sqrt{n}}{2}$ vectors L'_1, \dots, L'_t as the rows in a $t \times \frac{n}{2}$ matrix. Consider the top left $t \times t$ submatrix. Again, by Lemma 3.4, the probability of this matrix having rank at most $t - \frac{3\sqrt{n}}{2}$ is at most

$$2^{t - \left(t - \left(t - \frac{3\sqrt{n}}{2}\right)\right)^2} \leq 2^{\frac{n}{2} - \frac{9n}{4}} = 2^{-\frac{7n}{4}}$$

There are $2^n - 1$ choices of r , so by the union bound, the total probability of at least one of (3.7) or (3.8) failing for a least one choice is at most $2 \cdot (2^n - 1) \cdot 2^{-\frac{7n}{4}}$, which tends to zero, so in fact the described construction will have the desired nonlinearity with high probability. \square

We should note that it is not hard to improve in the constants in the proof and show that in fact the described function has nonlinearity at least $2^{n-1} - 2^{\frac{n}{2}+c\sqrt{n}}$ for some constant $c < 3$. However, the proof given does not allow improvement to e.g. $c = 2$.

It follows from the Gilbert-Varshamov bound (Theorem 1.7) implies that $L(n, \frac{n-1}{2}) < 2.95n$ for large enough n .

COROLLARY 3.9. *For sufficiently large n there exist (n, n) -functions with multiplicative complexity at most $2.95n$ with nonlinearity at least $2^{n-1} - 2^{\frac{n}{2}+3\sqrt{n}}$.*

4. Multiplicative Complexity of Vector Valued Functions

An $(n, 1)$ -function has multiplicative complexity between 0 and $(1 + o(1)) 2^{n/2}$, as shown in Nechiporuk (1962), (see also Jukna (2012)), and a random function has multiplicative complexity at least $2^{n/2} - O(n)$ (Boyar *et al.* 2000)

with probability $1 - o(1)$. However, there is no value of n where a concrete $(n, 1)$ -function has been exhibited with a proof that more than $n - 1$ AND gates are necessary to compute it. A lower bound of $n - 1$ follows by the simple *degree bound*¹: a function with degree d has multiplicative complexity at least $d - 1$ (Schnorr 1989).

In this section, we first show that repeated use of the degree bound is sufficient to prove that an explicit function has multiplicative complexity at least $2n - 3$.

Furthermore, we show that any (n, m) -function has multiplicative complexity at most $2.5(1 + o_m(1))\sqrt{m2^n}$ and that this is tight up to a small constant factor.

4.1. A Lower Bound for an Explicit Function. Here we show that repeated use of the degree bound gives a concrete (n, n) -function, exhibiting a lower bound of $2n - 3$. To the best of our knowledge this is the best lower bound on the multiplicative complexity for (n, n) -functions.

THEOREM 4.1. *The (n, n) -function f defined as $f_i(\mathbf{x}) = \prod_{j \in [n] \setminus \{i\}} x_j$, has multiplicative complexity at least $2n - 3$.*

PROOF. Consider the first AND gate, A , with degree at least $n - 1$. Such a gate exists since the outputs have degree $n - 1$. By the degree bound, A must have at least $p \geq n - 3$ AND gates with degree at most $n - 2$ in its subcircuit. Call these AND gates A_1, \dots, A_p . None of these AND gates can be an output gate. Suppose there are q additional AND gates (including A), where some of these must have degree at least $n - 1$. Call these AND gates B_1, \dots, B_q . Then, for every $i \in [n]$, there exist $P_i \subseteq [p]$, $Q_i \subseteq [q]$, and $X_i \subseteq [n]$ such that

$$f_i = \sum_{j \in P_i} A_j + \sum_{j \in Q_i} B_j + \sum_{j \in X_i} x_j.$$

We can think of each B_j (resp. A_j) as a vector in \mathbb{F}_2^n , where the i th coordinate is 1 if the term $\prod_{k \in [n] \setminus \{i\}} x_k$ is present in the algebraic normal form of the function computed by B_j (resp. A_j). Since each A_j has degree at most $n - 2$, all the A_j are zero vectors in this representation, so $\text{span}(A_1, \dots, A_p, B_1, \dots, B_q) = \text{span}(B_1, \dots, B_q)$. It follows that

$$\{f_1, \dots, f_n\} \subseteq \text{span}(B_1, \dots, B_q).$$

¹Notice that despite the name, this is not the same as Strassen's degree bound as described in Strassen (1973a) and Chapter 8 of Bürgisser *et al.* (1997).

Therefore,

$$n = \dim(\{f_1, \dots, f_n\}) \leq \dim(\text{span}(B_1, \dots, B_q)) \leq q.$$

We conclude that the circuit has at least $q + p \geq 2n - 3$ AND gates. \square

The multiplicative complexity of the function is at most $3n - 6$. This can be seen from the following construction:

1. Use $n - 2$ AND gates with the following outputs:

$$A = \{x_1x_2, x_1x_2x_3, \dots, x_1x_2 \dots x_{n-1}\}.$$

2. Use $n - 2$ AND gates with the following outputs:

$$B = \{x_2x_3 \dots x_n, x_3x_4 \dots x_n, \dots, x_{n-1}x_n\}.$$

3. Use $n - 4$ AND gates to AND together the i th element of A with the $i + 2$ nd element of B , for $1 \leq i \leq n - 4$.
4. Compute $x_1 \cdot (x_3x_4 \dots x_n)$ and $(x_1x_2 \dots x_{n-2}) \cdot x_n$.

We leave it as an interesting open question to close this gap.

4.2. Multiplicative Complexity of (n, m) -functions: Upper and Lower bounds. Below we look at the multiplicative complexity of the hardest (n, m) -functions. We give a construction showing that any such function has multiplicative complexity at most $2.5(1 + o_m(1))\sqrt{m2^n}$. For some values of n, m the construction gives the slightly better bound $2(1 + o_m(1))\sqrt{m2^n}$. A counting argument shows that this is at most a small factor from being tight.

THEOREM 4.2. *Let f be a random (n, m) -function, $m \leq 2^n$. Then, almost every f satisfies,*

$$c_\wedge(f) \geq \sqrt{m2^n} - 2n - \frac{m}{2}.$$

This proof is similar to the proof of Lemma 15 in Boyar *et al.* (2000).

PROOF. First we give an upper bound on the number of functions that can be computed with circuits using at most k AND gates. Let A_1, A_2, \dots, A_k be some topological ordering of the AND gates. The two inputs to A_i are the XORs of some of the previous AND gates, and some of the variables in

the circuit. Without loss of generality, we assume that the constant 1 is only used at the output gates of the circuit. Thus, the number of choices for A_i is

$$(2^{i-1+n})^2 / 2 = 2^{2i+2n-3}.$$

Each output gate of the circuit is the XOR of some of the AND gates, some of the variables to the circuit, and possibly the constant 1. That is, for $k \geq 0$ the total number of ways to choose the inputs to the AND gates and the outputs is:

$$(2^{k+n+1})^m \prod_{i=1}^k 2^{2i+2n-3} = 2^{k^2+m \cdot (1+n)+k \cdot (m+2n-2)}.$$

Since we assume that $m \leq 2^n$, we have that for sufficiently large n , $\sqrt{m2^n} - 2n - \frac{m}{2} > 0$ ($n \geq 12$ suffices). So the number of (n, m) -functions with multiplicative complexity at most $\sqrt{m2^n} - 2n - \frac{m}{2}$ is at most

$$2^{m2^n+2m-m^2/4+4n-\sqrt{m}2^{n/2+1}(1+n)}.$$

There are 2^{m2^n} different (n, m) -functions, so the probability that a random (n, m) -function can be computed with a circuit with at most $\sqrt{m2^n} - 2n - \frac{m}{2}$ AND gates is at most

$$2^{2m-m^2/4+4n-\sqrt{m}2^{n/2+1}(1+n)},$$

which tends to 0 when n goes to infinity. \square

On the other hand we present an almost matching upper bound. The technique has similarities to those used in (Lupanov 1958).

THEOREM 4.3. *Let f be an (n, m) -function. If $\log m$ is an integer and $n + \log m$ is even, then*

$$c_{\wedge}(f) \leq 2(1 + o_m(1))\sqrt{m2^n},$$

otherwise

$$c_{\wedge}(f) \leq 2.5(1 + o_m(1))\sqrt{m2^n}.$$

Before presenting the proof, we define indicator functions, and a result about their multiplicative complexity.

DEFINITION 4.4. *For every $n \in \mathbb{N}$ and $\mathbf{z} \in \mathbb{F}_2^n$ the indicator function $I_{\mathbf{z}}: \mathbb{F}_2^n \rightarrow \mathbb{F}_2$ is defined as $I_{\mathbf{z}}(\mathbf{x}) = 1$ if and only if $\mathbf{z} = \mathbf{x}$.*

The following simple proposition on the multiplicative complexity of indicator functions will be helpful in the proof.

PROPOSITION 4.5. Let $n > 1$ be arbitrary. Define the $(n, 2^n)$ -function

$$AI_n(\mathbf{x}) = (I_{(0,0,\dots,0)}(\mathbf{x}), I_{(1,0,\dots,0)}(\mathbf{x}), \dots, I_{(1,1,\dots,1)}(\mathbf{x})).$$

Then the multiplicative complexity of AI_n is $c_\wedge(AI_n) = 2^n - n - 1$

PROOF. First, we show that $c_\wedge(AI_n) \leq 2^n - n - 1$. We start by computing all quadratic terms, that is terms on the form $x_i x_j$ for $1 \leq i < j \leq n$. This can be done using one AND gate for each of the $\binom{n}{2}$ terms. Now we compute each degree three term. Since each degree three term $x_i x_j x_k$ can be written as $x_i Q$ for some quadratic term Q , we can do this with one AND gate for each term. We continue in this way until we have computed the term $\prod_{i \in S} x_i$ for each $S \subseteq [n]$. The number of AND gates used is

$$\binom{n}{2} + \binom{n}{3} + \dots + \binom{n}{n-1} + \binom{n}{n} = 2^n - n - 1.$$

Now that all the terms that can occur in an ANF have been obtained, any function can be computed without using additional AND gates. In particular, all the indicator functions can be computed using no additional AND gates.

For the lower bound, suppose M AND gates, A_1, \dots, A_M , suffice to compute the 2^n indicator functions. Let T_1, \dots, T_{2^n} be some ordering of the terms $\prod_{i \in S} x_i$, $S \subseteq [n]$. Each function $f: \mathbb{F}_2^n \rightarrow \mathbb{F}_2$ can be considered as a vector in $\mathbb{F}_2^{2^n}$ by letting the i th coordinate be 1 if and only if term T_i is included in the ANF of f . Considering all functions as vectors in this way, it follows that

$$\{I_{\mathbf{z}} | \mathbf{z} \in \mathbb{F}_2^n\} \subseteq \text{span}_{\mathbb{F}_2}\{1, x_1, \dots, x_n, A_1, \dots, A_M\}.$$

All the indicator functions are linearly independent, so we have

$$2^n = \dim(\{I_{\mathbf{z}} | \mathbf{z} \in \mathbb{F}_2^n\}) \leq \dim(\text{span}_{\mathbb{F}_2}\{1, x_1, \dots, x_n, A_1, \dots, A_M\}) \leq M + n + 1.$$

□

PROOF (Proof of Theorem 4.3.). In the following let k be an integer to be determined later. First compute all the indicator functions on the last $n - k$ variables $x_{k+1}, x_{k+2}, \dots, x_n$. This uses $2^{n-k} - (n - k) - 1$ AND gates. Given all these indicators, using only XOR gates, it is possible to compute the function

$$f_i(a_1, a_2, \dots, a_k, x_{k+1}, x_{k+2}, \dots, x_n),$$

for any choice of constants $i \in [m]$ and $\mathbf{a} \in \mathbb{F}_2^k$. Now compute all the indicator functions on the first k variables using $2^k - k - 1$ AND gates. After this, for each $i \in [m]$, and for each $\mathbf{a} \in \mathbb{F}_2^k$ compute

$$g_{i,\mathbf{a}}(\mathbf{x}) := I_{(a_1,\dots,a_k)}(x_1, \dots, x_k) \cdot f_i(a_1, a_2, \dots, a_k, x_{k+1}, x_{k+2}, \dots, x_n).$$

This uses $m \cdot 2^k$ AND gates. Now observe that

$$f_i(\mathbf{x}) = \sum_{\mathbf{a} \in \mathbb{F}_2^k} g_{i,\mathbf{a}}(\mathbf{x}),$$

so each f_i can be obtained using only XOR gates. The total number of AND gates used is less than

$$m2^k + 2^{n-k} + 2^k.$$

Suppose that $\log m$ is an integer and $n + \log m$ is even. Letting $k = \frac{n - \log m}{2}$ results in

$$m2^{\frac{n - \log m}{2}} + 2^{\frac{n + \log m}{2}} + 2^{\frac{n - \log m}{2}} = (1 + o_m(1))2\sqrt{m}2^{\frac{n}{2}}.$$

Otherwise we let $k = \lceil \frac{n - \log m}{2} \rceil$. Let $k = \frac{n - \log m}{2} + \epsilon$. Then the total number of AND gates is at most

$$(1 + o_m(1))\sqrt{m}2^{\frac{n}{2}}(2^\epsilon + 2^{-\epsilon}) \leq 2.5(1 + o_m(1))\sqrt{m}2^{\frac{n}{2}}.$$

□

5. Open Problems

Strassen (Strassen 1973b) (see also Bürgisser *et al.* (1997), Proposition 14.1, p. 351) proved that for an *infinite field*, \mathbb{K} , if the quadratic function $F: \mathbb{K}^n \rightarrow \mathbb{K}^m$ can be computed with M multiplications/divisions, then it can be computed in M multiplications by a quadratic circuit. However, it is unknown whether a similar result holds for *finite fields* in particular for \mathbb{F}_2 . Mirwald and Schnorr (Mirwald & Schnorr 1992) showed that for quadratic $(n, 1)$ - and $(n, 2)$ -functions, quadratic circuits are optimal. It is still not known whether this is true for (n, m) -functions in general. It would be very interesting to determine if the bounds proven here for quadratic circuits also hold for general circuits.

When inspecting the proof of Theorem 3.1, one can make a weaker assumption on the circuit than it having $\Sigma\Pi\Sigma$ structure. For example, it is sufficient if it holds that for every AND gate, A , there is a unique AND gate, A' (which might be equal to A), such that every path from A to an output goes through A' . Can one find a larger, interesting class of circuits where the proof holds?

The function defined in Theorem 4.1 has multiplicative complexity at least $2n - 3$ and at most $3n - 6$. What is the exact value?

Acknowledgements

This work was partially supported by the Danish Council for Independent Research, Natural Sciences. Most of the results in this paper appeared in (Boyar & Find 2014).

References

- MARTIN R. ALBRECHT, CHRISTIAN RECHBERGER, THOMAS SCHNEIDER, TYGE TIESSEN & MICHAEL ZOHNER (2015). Ciphers for MPC and FHE. In *Advances in Cryptology: Proceedings of EUROCRYPT 2015*, Sofia, Bulgaria, ELISABETH OSWALD & MARC FISCHLIN, editors, volume 9056 of *Lecture Notes in Computer Science*, 430–454. Springer.
- JOAN BOYAR, MAGNUS FIND & RENÉ PERALTA (2013). Four Measures of Non-linearity. In *Proc. 8th International Conf. on Algorithms and Complexity (CIAC)*, PAUL G. SPIRAKIS & MARIA J. SERNA, editors, volume 7878 of *Lecture Notes in Computer Science*, 61–72. Springer. Eprint with correction available at the Cryptology ePrint Archive, Report 2013/633, 2013, <http://eprint.iacr.org/> .
- JOAN BOYAR & MAGNUS GAUSDAL FIND (2014). The Relationship between multiplicative complexity and nonlinearity. In *Mathematical Foundations of Computer Science 2014 - 39th International Symposium, MFCS 2014, Budapest, Hungary, August 25-29, 2014. Proceedings, Part II*, ERZSÉBET CSUHAJ-VARJÚ, MARTIN DIETZFELBINGER & ZOLTÁN ÉSIK, editors, volume 8635 of *Lecture Notes in Computer Science*, 130–140. Springer.
- JOAN BOYAR & RENÉ PERALTA (2008). Tight bounds for the multiplicative complexity of symmetric functions. *Theor. Comput. Sci.* **396**(1-3), 223–246.
- JOAN BOYAR, RENÉ PERALTA & DENIS POCHUEV (2000). On the multiplicative complexity of Boolean functions over the basis $(\wedge, \oplus, 1)$. *Theor. Comput. Sci.* **235**(1), 43–57.
- MARK R. BROWN & DAVID P. DOBKIN (1980). An Improved Lower Bound on Polynomial Multiplication. *IEEE Trans. Computers* **29**(5), 337–340.
- NADER H. BSHOUTY & MICHAEL KAMINSKI (2006). Polynomial multiplication over finite fields: from quadratic to straight-line complexity. *Computational Complexity* **15**(3), 252–262.
- PETER BÜRGISSER, MICHAEL CLAUSEN & MOHAMMAD AMIN SHOKROLLAHI (1997). *Algebraic Complexity Theory*, volume 315 of *Grundlehren der mathematischen Wissenschaften*. Springer, I-XXIII, 1-618 .

CLAUDE CARLET (2002). On the complexity of cryptographic Boolean functions. In *6th Conference on Finite Fields and Applications*, G.L. MULLEN, H. STICHTENOTH & EDS. H. TAPIA-RECILLAS, editors, 53–69. Springer.

CLAUDE CARLET (2010a). Boolean functions for cryptography and error correcting codes. In *Boolean Models and Methods in Mathematics, Computer Science, and Engineering*, YVES CRAMA & PETER L. HAMMER, editors, chapter 8, 257–397. Cambridge, UK: Cambridge Univ. Press.

CLAUDE CARLET (2010b). Vectorial Boolean Functions for Cryptography. In *Boolean Models and Methods in Mathematics, Computer Science, and Engineering*, YVES CRAMA & PETER L. HAMMER, editors, chapter 9, 398–469. Cambridge, UK: Cambridge Univ. Press.

FLORENT CHABAUD & SERGE VAUDENAY (1995). Links Between Differential and Linear Cryptanalysis. In *Advances in Cryptology: Proceedings of EUROCRYPT 1994*, Perugia, Italy, ALFREDO DE SANTIS, editor, volume 950 of *Lecture Notes in Computer Science*, 356–365. Springer.

JOAN DAEMEN & VINCENT RIJMEN (2002). *The Design of Rijndael: AES-The Advanced Encryption Standard*. Security and Cryptology. Springer.

YARKIN DORÖZ, ARIA SHAHVERDI, THOMAS EISENBARTH & BERK SUNAR (2014). Toward Practical Homomorphic Evaluation of Block Ciphers Using Prince. In *Financial Cryptography and Data Security - FC 2014 Workshops, BITCOIN and WAHC 2014*, Christ Church, Barbados, RAINER BÖHME, MICHAEL BRENNER, TYLER MOORE & MATTHEW SMITH, editors, volume 8438 of *Lecture Notes in Computer Science*, 208–220. Springer.

STASYS JUKNA (2011). *Extremal Combinatorics: with Applications in Computer Science*. Texts in Theoretical Computer Science. Springer, 2nd edition.

STASYS JUKNA (2012). *Boolean Function Complexity: Advances and Frontiers*. Springer Berlin Heidelberg.

MICHAEL KAMINSKI & NADER H. BSHOUTY (1989). Multiplicative complexity of polynomial multiplication over finite fields. *J. ACM* **36**(1), 150–170.

VLADIMIR KOLESNIKOV & THOMAS SCHNEIDER (2008). Improved Garbled Circuit: Free XOR Gates and Applications. In *Proceedings of the 35th International Colloquium on Automata, Languages and Programming ICALP 2008*, Reykjavik, Iceland, LUCA ACETO, IVAN DAMGÅRD, LESLIE ANN GOLDBERG, MAGNÚS M. HALLDÓRSSON, ANNA INGÓLFSDÓTTIR & IGOR WALUKIEWICZ, editors, volume 5126 of *Lecture Notes in Computer Science*, 486–498. Springer.

ILAN KOMARGODSKI, RAN RAZ & AVISHAY TAL (2013). Improved Average-Case Lower Bounds for DeMorgan Formula Size. In *Proceedings of the 54th Annual IEEE Symposium on Foundations of Computer Science*, Berkeley, CA, 588–597.

ABRAHAM LEMPEL, GADIEL SEROUSSI & SHMUEL WINOGRAD (1983). On the Complexity of Multiplication in Finite Fields. *Theor. Comput. Sci.* **22**, 285–296.

TANCRÈDE LEPOINT & PASCAL PAILLIER (2013). On the Minimal Number of Bootstrappings in Homomorphic Circuits. In *Financial Cryptography and Data Security - FC 2013 Workshops, USEC and WAHC 2013, Okinawa, Japan*, ANDREW A. ADAMS, MICHAEL BRENNER & MATTHEW SMITH, editors, volume 7862 of *Lecture Notes in Computer Science*, 189–200. Springer.

OLEG B LUPANOV (1958). A method of circuit synthesis. *Izv. VUZ Radiofiz* **1**(1), 120–140.

ROBERT J. McELIECE, EUGENE R. RODEMICH, HOWARD RUMSEY JR. & LLOYD R. WELCH (1977). New upper bounds on the rate of a code via the Delsarte-MacWilliams inequalities. *IEEE Trans. Inform. Theory* **23**(2), 157–166.

ROLAND MIRWALD & CLAUS-PETER SCHNORR (1992). The Multiplicative Complexity of Quadratic Boolean Forms. *Theor. Comput. Sci.* **102**(2), 307–328.

E. I. NECHIPORUK (1962). On the complexity of schemes in some bases containing nontrivial elements with zero weights (In Russian). *Problemy Kibernetiki* **8**, 123–160.

ALEXANDER A RAZBOROV (1987). Lower bounds on the size of bounded depth circuits over a complete basis with logical addition. *Mathematical Notes* **41**(4), 333–338.

CLAUS-PETER SCHNORR (1989). The Multiplicative Complexity of Boolean Functions. In *Algebraic Algorithms and Error-Correcting Codes: AAECC-6*, Rome, Italy, 1988, TEO MORA, editor, volume 357 of *Lecture Notes in Computer Science*, 45–58. Springer, Heidelberg.

N.J.A. SLOANE & F.J. MACWILLIAMS (1977). The Theory of Error Correcting Codes. *North-Holland Math. Library* **16**.

VOLKER STRASSEN (1973a). Die Berechnungskomplexität von elementarsymmetrischen Funktionen und von Interpolationskoeffizienten. *Numerische Mathematik* **20**(3), 238–251.

VOLKER STRASSEN (1973b). Vermeidung von Divisionen. *Journal für die reine und angewandte Mathematik* **264**, 184–202.

VINOD VAIKUNTANATHAN (2011). Computing Blindfolded: New Developments in Fully Homomorphic Encryption. In *Proceedings of the 52th Annual IEEE Symposium on Foundations of Computer Science*, Palm Springs, CA, RAFAIL OSTROVSKY, editor, 5–16. IEEE.

YULIANG ZHENG, XIAN-MO ZHANG & HIDEKI IMAI (1999). Restriction, Terms and Nonlinearity of Boolean Functions. *Theor. Comput. Sci.* **226**(1-2), 207–223.

MAGNUS GAUSDAL FIND
Information Technology Laboratories
National Institute of Standards and
Technology
Maryland, USA
`magnus.find@nist.gov`

JOAN BOYAR
Department of Mathematics and Com-
puter Science
University of Southern Denmark
Odense, Denmark
`joan@imada.sdu.dk`