Physical basis for half-integral Shapiro steps in a DC SQUID

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Abstract

The dynamics of a DC SQUID is analogous to the classical dynamics of a particle subject to conservative, damping, and driving forces in two dimensions. The equations of motion define a trajectory on a potential-energy surface derived from the conservative forces, the components of which correspond to different forms of stored energy in the SQUID. In the presence of a periodic driving force, half-integral Shapiro steps are possible when the trajectory follows a zig-zag path between minima of the potential surface. This description of the dynamics in terms of a potential surface provides an intuitive, physical basis for previous simulation results on half-integral Shapiro steps in a DC SQUID.

1. Introduction

Half-integral Shapiro steps were recently observed in high-\(T_c\) grain-boundary junctions [1], as well as in step-edge junctions [2]. Similar steps were also present in a DC superconducting quantum interference device (SQUID) subjected to an external magnetic field and microwave radiation [3]. This prompted a numerical simulation of the response of a parallel array of Josephson junctions to applied microwave radiation [3]. This model predicted a dependence of the Shapiro step width on microwave amplitude for both integral and half-integral steps. Excellent agreement between simulation and experimental results provided quantitative evidence that high-\(T_c\) grainboundary junctions are composed of several junctions in parallel, only two or three of which have relatively large critical currents [4]. These junctions dominate in measurements of the dependence of the step width on microwave amplitude. Thus, many grain-boundary junctions are equivalent to two junctions in parallel, a DC SQUID.

While there has been much interest in DC SQUID's as sensitive magnetometers [5], comparatively little is known about their response to microwave radiation and the occurrence of half-integral Shapiro steps. Vanneste et al. [3] were the first to observe half-integral steps in a DC SQUID, for which they presented a qualitative physical explanation. The response of a DC SQUID to microwaves has also been used to explain subharmonic steps in two-dimensional arrays of Josephson junctions [6]. We make the explanation of half-integral steps in a DC SQUID more intuitive by using a potential-energy surface to describe the dynamics of the phases across the junctions. This type of description has been used previously for thermal noise [7] and memory cell operation [8] of a DC SQUID and for an underdamped two-dimensional Josephson junction.
array [9]. Our purpose here is to present a physical basis for previous simulation results on half-integral steps [4], thereby providing insight into the response of a current-driven DC SQUID subjected to applied magnetic fields.

2. Potential-energy surface

A potential-energy surface is derived from the equations describing two parallel junctions, with the phases across the junctions as the two independent variables. The trajectory of the phases on the potential surface is shown to be analogous to the classical motion of a particle on this surface subject to both damping and driving forces. Each term in the potential is shown to follow from basic energy considerations, and a method for using the potential to calculate the magnetic-field dependence of the critical current is presented.

2.1. Equations for a DC SQUID

A DC SQUID is a circuit consisting of two Josephson junctions in parallel, here indexed by $j = 0$ and 1. Each junction has a phase difference $\delta_j$ across it and is described by the resistively shunted-junction (RSJ) model [10,11], with a sinusoidal current–phase relation and a shunting resistance. The junctions are assumed to be overdamped so there is no parallel capacitance. The total critical current is $I_c$ and the total normal-state resistance is $R_n$. The critical current of each junction is $I_{c,j}$; thus $\eta_j$ is the fraction of the total critical current carried by junction $j$ and $\eta_0 + \eta_1 = 1$. The resistance of each junction is $R_j$, and $R_j = R_n/\eta_j$. Since the two junctions are of the same type, it is implied that both $I_{c,j}R_j$ products are equal to $I_cR_n$ [12]. The total inductance of the loop formed by the DC SQUID is $L$, which is lumped into circuit elements $L_j$, and a magnetic flux $\Phi$ is enclosed by the loop. For simplicity, only the symmetric case $L_0 = L_1 = L/2$ is considered in this paper. The area of the loop is assumed to be much greater than the areas of the junctions perpendicular to the applied magnetic field.

The equations describing the dynamics of a DC SQUID subjected to microwave radiation and an external magnetic field are as follows [3,13,14]. The DC SQUID can have a DC current bias, and microwave radiation is equivalent to an AC current source. This is expressed by a current $I$ passing through the DC SQUID with both DC and AC components, given by

$$I = I_{DC} + I_{AC} \sin(\omega t).$$

Here, the current amplitudes are given by $I_{DC}$ and $I_{AC}$ and $\omega$ is the angular frequency of the applied microwave field. The current $I_j$ through a single junction is, from current conservation through the Josephson element and the resistor,

$$I_j = \frac{V_j}{R_j} + I_{c,j} \sin \delta_j.$$

The voltage $V_j$ across a junction is given by the fundamental Josephson relation [15]

$$V_j = \frac{h}{2e} \frac{d\delta_j}{dt} = \frac{\Phi_0}{2\pi} \frac{d\delta_j}{dt}.$$

The difference between the phases due to the vector potential of the enclosed magnetic flux is

$$\delta_1 - \delta_0 = 2\pi \frac{\Phi}{\Phi_0},$$

where $\Phi_0 = h/2e$ is the flux quantum. The enclosed magnetic flux is divided between an externally applied flux $\Phi_0$ and the flux resulting from the circulating current in the loop, yielding

$$\Phi = \Phi_0 + \frac{L}{2}(I_0 - I_1).$$

Converting to dimensionless units, a normalized time $\tau = \omega_0 t$ and frequency $\Omega = \omega/\omega_0$ are defined where $\omega_0 = (2e/h)L/R_n$, as well as a dimensionless parameter $\beta_c = (2\pi/\Phi_0)Lc$.

Using current conservation of the DC SQUID, Eqs. (1) to (3) reduce to

$$i_{DC} + i_{AC} \sin(\Omega \tau) = \frac{d(\eta_0 \delta_0)}{d\tau} + \frac{d(\eta_1 \delta_1)}{d\tau} + \eta_0 \sin \delta_0 + \eta_1 \sin \delta_1,$$

(6)
where the current amplitudes have been normalized by \( I_c \). Combining Eqs. (4) and (5) gives

\[
\frac{2}{\beta_L} (\delta_1 - \delta_0 - 2\pi f_s) = \frac{d(\eta_0 \delta_0)}{d\tau} - \frac{d(\eta_1 \delta_1)}{d\tau} + \eta_0 \sin \delta_0 - \eta_1 \sin \delta_1 ,
\]

where \( f_s = \Phi_s / \Phi_0 \) is the normalized applied magnetic flux. Solving for the derivative terms yields

\[
\frac{d(\eta_0 \delta_0)}{d\tau} = \frac{1}{2} (i_{\text{DC}} + i_{\text{AC}} \sin (\Omega \tau)) - \eta_0 \sin \delta_0 + \frac{1}{\beta_L} (\delta_1 - \delta_0 - 2\pi f_s),
\]

\[
\frac{d(\eta_1 \delta_1)}{d\tau} = \frac{1}{2} (i_{\text{DC}} + i_{\text{AC}} \sin (\Omega \tau)) - \eta_1 \sin \delta_1 - \frac{1}{\beta_L} (\delta_1 - \delta_0 - 2\pi f_s).
\]

These are the equations describing the dynamics of a DC SQUID with both an applied magnetic field and microwave radiation.

2.2. Justification for the potential

It is first useful to motivate and illuminate deriving a potential from Eqs. (8) and (9) by making an analogy with classical mechanics. A pendulum is one mechanical analog of a Josephson junction described by the RSJ model and has received considerable attention \([16,17]\). A better analog for our purpose is that of a particle moving on a potential energy surface \([17-19]\). For a single Josephson junction with phase difference \( \delta \), critical current \( I_c \), normal-state resistance \( R_n \), and capacitance \( C \), the dynamics are analogous to a particle with mass proportional to \((h/2e)C\) on a tilted cosine (or washboard) potential \( U_s = -(\Phi_0/2\pi) I_c \cos \delta + I_{\text{DC}} \delta \) in a viscous fluid with a damping coefficient proportional to \((h/2e)/R_c\). Here, \( \delta \) is the generalized coordinate and corresponds to the position of the particle on the washboard potential. With no driving force, the particle is subjected to two forces, one from the gradient of the potential and the other from damping.

Since the junctions of the DC SQUID we are considering here have no capacitance, the mass of the analogous particle is zero, and therefore the sum of all the forces is also zero. Separating the forces according to their origins yields

\[
F_{\text{cons}} + F_{\text{damp}} + F_{\text{drive}} = 0 ,
\]

where the conservative force \( F_{\text{cons}} = -\nabla U \), \( U \) is a potential, and there are damping and driving forces \( F_{\text{damp}} \) and \( F_{\text{drive}} \), respectively. Using Eq. (3), the dissipation function \( \Im \) due to current flow through the resistors is given by

\[
\Im = \frac{1}{2} \sum_{j=0}^{\infty} \left( \frac{V_j}{R_j} \right)^2
\]

\[
= \frac{1}{2} \left( \frac{\Phi_0}{2\pi} \right)^2 \sum_{j=0}^{\infty} \frac{1}{R_j} \left( \frac{d\delta_j}{dt} \right)^2.
\]

Now, \( F_{\text{damp}} = -\nabla_{\delta} \Im \), so that

\[
F_{\text{damp}} = -\frac{i_{\text{AC}} I_c}{2\pi} \sum_{j=0}^{\infty} \frac{d(\eta_j \delta_j)}{d\tau} \delta_j.
\]

Comparing Eq. (12) with the sum of Eqs. (8) and (9) and identifying \( -\nabla U \) with the time-independent terms and \( f_{\text{drive}} \) with the time-dependent terms gives

\[
f_{\text{drive}} = \frac{i_{\text{AC}}}{2} (\delta_0 + \delta_1) \sin (\Omega \tau) (\delta_0 + \delta_1).
\]

Thus, the driving force is simply the applied AC current. Finally, by integrating the time-independent terms of Eqs. (8) and (9) by \( \delta_0 \) and \( \delta_1 \), respectively, the normalized potential \( u \) is given by

\[
u = -\frac{i_{\text{DC}}}{2} (\delta_0 + \delta_1) - \frac{\eta_0 \cos \delta_0 - \eta_1 \cos \delta_1}{2\beta_L} (\delta_1 - \delta_0 - 2\pi f_s)^2.
\]
\[ \beta_t = (2c/\hbar)I_c R_e^2 C, \]

then there is an additional term on the left-hand side of Eq. (12) given by
\[ \beta_t \sum_{j=0}^{1} \frac{d^2(\eta_j \delta_j)}{d\tau^2} \delta_j. \]

This is equivalent to a non-zero mass of the analogous particle. Since this additional term does not affect the form of the potential, Eq. (13) applies to DC SQUID's regardless of the capacitance of the junctions.

2.3. Components of the potential

The first term on the right-hand side of Eq. (13) is the potential of a constant force, from the applied DC current, along the diagonal in the \( \delta_0 - \delta_1 \) plane. The second and third terms are the normalized energy stored in the junctions, as derivable from Ref. [20]. As for the fourth term, \( \delta_0 - \delta_0 - 2\pi f_s = \beta_t (\Delta i/2) \), where \( \Delta i = i_0 - i_1 \) is the circulating current in the DC SQUID. Thus, the fourth term is \( (\beta_t/2)(\Delta i/2)^2 \), which upon multiplying by \( \Phi_o I_c/2\pi \) becomes \( (L/2)(\Delta i/2)^2 \). This is simply the magnetic field energy from a current \( \Delta i/2 \) through an inductance \( L \). Therefore, Eq. (12) can also be thought of as an energy balance between energies stored in the inductor and junctions, dissipated in the resistors, and supplied by the current.

A potential surface with independent coordinates \( \delta_0 \) and \( \delta_1 \) is described by Eq. (13). In order to understand and visualize this potential, it is convenient to divide it into three basic components. The most important one comes from the second and third terms, which is a double cosine surface shown in Fig. 1 as a contour plot in the \( \delta_0 - \delta_1 \) plane for two cases: (a) equal junctions, or \( \eta_0 = \eta_1 = \frac{1}{2} \), and (b) unequal junctions with \( \eta_0 = \frac{1}{3} \). The insets are three-dimensional plots of these surfaces. The heights of the barriers due to maxima are independent of \( \eta_0 \), while the heights of those due to saddle points depend on both \( \eta_0 \) and \( \eta_1 \). Thus, as illustrated in Fig. 1(b), for \( \eta_0 > \eta_1 \), the barrier heights of the saddle points are smaller in the \( \delta_0 \) direction (e.g., between points a and b) than in the \( \delta_1 \) direction (between points b and c).

The fourth term in the potential is a parabolic surface with its trough running along the line \( \delta_1 = \delta_0 + 2\pi f_s \). This line is shown in Fig. 1(a) for \( f_s = 0 \).
\[ i_2, \text{ and } 1, \text{ while Fig. 1(c) is a three-dimensional plot of the potential including both the double cosine and the parabolic surfaces for equal junctions with } \beta_L = 10 \text{ and } f_s = 0. \text{ The steepness of the parabolic surface increases with decreasing } \beta_L. \text{ Finally, the first term in Eq. (13) causes a downward tilt of the potential surface in the diagonal direction. Note that if the two arms of the DC SQUID are asymmetric, meaning the inductances } L_0 \text{ and } L_1 \text{ are not equal, then this tilt is at an angle to the diagonal direction, determined by the amount of asymmetry.}

To summarize, the three basic components of the potential surface from Eq. (13) are a double cosine surface, a parabolic surface with its trough along a diagonal, and a negative tilt of the surface in the diagonal direction. The dynamics of the phases } \delta_0 \text{ and } \delta_1 \text{ describe a trajectory on this potential surface, which is analogous to a particle of zero mass moving on this surface subject to the forces } f_{\text{damp}} \text{ and } f_{\text{drive}}.

2.4. Magnetic-field dependence of the critical current

The potential in Eq. (13) can be used to calculate the magnetic-field dependence of the critical current of a DC SQUID, } i_c(f_s) \text{ in normalized units. Returning to a single junction, the normalized potential is } \mu_s = -i_{\text{DC}} \delta - \cos \delta, \text{ where } i_{\text{DC}} = I_{\text{DC}}/I_c. \text{ With no external driving force, the phase } \delta \text{ will be stationary in a minimum, so by Eq. (3) there will be no voltage across the junction. The condition for a minima in the potential is } \sin \delta = i_{\text{DC}}, \text{ which has solutions only for } i_{\text{DC}} \leq 1. \text{ For } i_{\text{DC}} > 1 \text{ there are no minima, so the phase increases with time. Thus there is a voltage across the junction, implying that the critical current has been exceeded. Therefore, in normalized units, } i_c = 1 \text{ as expected for a single junction.}

For a DC SQUID, two equations must be solved simultaneously for there to be a minimum, namely

\[
\frac{\partial \mu}{\partial \delta_0} = -\frac{i_{\text{DC}}}{2} + \eta_0 \sin \delta_0 - \frac{1}{\beta_L} (\delta_1 - \delta_0 - 2\pi f_s) = 0, \tag{14}
\]

\[
\frac{\partial \mu}{\partial \delta_1} = -\frac{i_{\text{DC}}}{2} + \eta_1 \sin \delta_1 + \frac{1}{\beta_L} (\delta_1 - \delta_0 - 2\pi f_s) = 0. \tag{15}
\]

Following Tesche [7], } \delta_i( \delta_0 ) \text{ and } \delta_0( \delta_1 ) \text{ are obtained from Eqs. (14) and (15), respectively, to yield

\[
\delta_i( \delta_0 ) = \delta_0 + \beta_L \left( -\frac{i_{\text{DC}}}{2} + \eta_0 \sin \delta_0 \right) + 2\pi f_s, \tag{16}
\]

\[
\delta_0( \delta_1 ) = \delta_1 + \beta_L \left( -\frac{i_{\text{DC}}}{2} + \eta_1 \sin \delta_1 \right) - 2\pi f_s. \tag{17}
\]

To obtain } i_c(f_s) \text{, Eqs. (16) and (17) are solved simultaneously at a fixed value of } f_s \text{ with increasing values of } i_{\text{DC}}. \text{ The greatest value of } i_{\text{DC}} \text{ for which there is a solution is } i_c. \text{ This criteria for two junctions is entirely equivalent to that for a single junction discussed in the preceding paragraph. This procedure is repeated for increasing values of } f_s. \text{ Obviously, } i_c(f_s) \text{ now depends on } \beta_L \text{ and } \eta_0, \text{ as indicated by other authors [13,14]. Results for } i_c(f_s) \text{ obtained with this procedure agree with those obtained by searching for stable solutions of Eqs. (8) and (9) with no AC current, as detailed in Ref. [4].}

It is well-known that for a DC SQUID the maximum modulation of } i_c \text{ with } f_s \text{ depends on } \beta_L \text{ [21]. For equal junctions this dependence on } \beta_L \text{ can be explained using the potential surface. For } f_s = 0.5, \text{ corresponding to the maximum modulation of } i_c, \text{ the surface is symmetric about the diagonal line } \delta_1 = \delta_0 + \pi, \text{ along which the potential is constant. For small } \beta_L, \text{ the surface has a very narrow trough along this line, confining the trajectory to a region with small barrier heights. Thus, } i_c \text{ is nearly zero, and is exactly zero in the limit of no inductance. For large } \beta_L, \text{ the surface has a wide trough, so the trajectory can easily reach a minimum. Since the critical current required to escape from this minimum is approximately the same for all values of } f_s, \text{ } i_c \text{ is only slightly less than one.}

3. Dynamics

The potential-energy surface derived in the preceding section is used to explain the occurrence and behavior of half-integral Shapiro steps in a DC SQUID. A brief review of simulations of step width
as a function of AC current amplitude, $\Delta i_{DC}(i_{AC})$, is given, followed by calculations of the position of the minima of the surface. The important features of the potential surface which determine the trajectories of the phases, and therefore the occurrence and behavior of half-integral steps, are described for both equal and unequal junctions.

3.1. Review of previous results

When an AC current is applied to a Josephson junction, constant voltage, or Shapiro [22], steps are observed in the current–voltage curve of the junction. The voltages of these steps are given by $V = (h/2e)n\nu$, where $\nu$ is the frequency of the AC current and $n$ indexes the step number and is integral for a single junction without capacitance [23,24]. In a DC SQUID, there can also be additional Shapiro steps indexed by half-integral $n$. Using the RSJ description of the junctions, the normalized width $\Delta i_{DC}$ of all Shapiro steps are smooth functions of the normalized AC current amplitude $i_{AC}$ for both a single junction [25] and a DC SQUID [4].

Simulation results were obtained by numerically solving Eqs. (8) and (9) using the fourth-order Runge–Kutta method with a step size of 0.01(\omega_0). Plots of $\Delta i_{DC}(i_{AC})$ for fixed $\Omega$, $\beta_0$, $\eta_0$, and $f_a$ were generated by finding the limits of $i_{DC}$ for which $\eta_0 S_0 + \eta_1 S_1$ advances by an average of $2\pi n$ in two AC cycles for a fixed value of $i_{AC}$, following a previous suggestion [26] and implementation [27] of this procedure. The width of step $n$ at that value of $i_{AC}$ is then the difference between the minimum and maximum values of $i_{DC}$, and this procedure is repeated for different values of $n$ and $i_{AC}$. The same technique was used for the dependence of the normalized critical current on normalized magnetic field, $i_c(f_a)$, for which $n = 0$, $i_{AC} = 0$, and $f_a$ is varied.

Simulation results of $\Delta i_{DC}(i_{AC})$ for $n = 0$ to 2 of a DC SQUID with $\Omega = 0.175$, $\beta_0 = 10$, and different values of the applied flux $f_a$ are shown in Fig. 2 for ((a) and (b)) equal and ((c) to (f)) unequal junctions. Note that only the positive current polarity is used for $\Delta i_{DC}$ of the $n = 0$ step. These plots illustrate the different behaviors of $\Delta i_{DC}(i_{AC})$ for both integral and half-integral steps given details of in Ref. [4]. For equal junctions, $i_c$ is a maximum for $f_a = 0$ and a minimum for $f_a = 0.5$, corresponding to Figs. 2(a) and (b), respectively. Also, $i_c(f_a)$ is symmetric about $f_a = 0.5$. Briefly, for $f_a = 0$ there are no half-integral steps and the integral steps have RSJ behavior, while for $f_a = 0.5$ the integral steps have non-RSJ behavior and the half-integral steps have their maximum width and a coincident behavior (maxima in $\Delta i_{DC}$ for both the $n = \frac{1}{2}$ and $\frac{3}{2}$ steps occur at the same values of $i_{AC}$). Also, varying the value of $\Omega$ by a factor of two has little effect on these behaviors; the widths of the half-integral steps increase with decreasing $i_c$, and the range of $f_a$ over which half-integral steps occur increases with increasing $\beta_0$. For unequal junctions, the values of $f_a$ corresponding to the maximum and minimum of $i_c$ are shifted from those for equal junctions to 0.7 and
0.2, respectively, as shown in Figs. 2(f) and (d). There are now half-integral steps for \( f_a = 0 \) (Fig. 2(c)); the half-integral steps are coincident for \( f_a = 0.5 \) (Fig. 2(e)) but paired and alternating for \( f_a = 0.2 \) (with increasing \( i_{AC} \), there are two maxima in \( \Delta i_{DC} \) for the \( n = \frac{1}{2} \) step, followed by two maxima for the \( n = \frac{3}{2} \) step, etc.), and in all cases the integral steps have a modified RSJ behavior (local minima in \( \Delta i_{DC} \)). Also, the widths of the half-integral steps still increase with decreasing \( i_c \).

3.2. Location of minima of the potential surface

In the following, it is convenient to know the locations of minima of the potential surface. In order to obtain analytical expressions for these locations, let \( \delta_0 = 2\pi l + x \) and \( \delta_1 = 2\pi m + y \), where \( l \) and \( m \) are integers and \( x \) and \( y \) are small deviations from the minima of the double cosine surface, i.e. \( i_{DC} \ll i_c \). Equations for \( x \) and \( y \) are obtained by simultaneously solving Eqs. (14) and (15) with the expressions for \( \delta_0 \) and \( \delta_1 \) from above and linearizing the sine terms. There are two important lines of minima of the potential surface, those near the diagonal lines \( \delta_1 = \delta_0 \) and \( \delta_1 = \delta_0 + 2\pi \), corresponding to \( m = l \) and \( m = l + 1 \), respectively. These are the two lines defined, respectively, by the points a, c, and e and by b, d, and f in Fig. 1. We restrict consideration of the potential surfaces to those for which \( 0 \leq f_a \leq 1 \) since potential surfaces with other values of \( f_a \) are equivalent to these. The results are, for \( \delta_1 = \delta_0 \),

\[
\begin{align*}
x &= \frac{-2\pi \eta_1 f_a + i_{DC}(1 + \beta_1 \eta_1/2)}{1 + \beta_1 \eta_0 \eta_1}, \\
y &= \frac{2\pi \eta_0 f_a + i_{DC}(1 + \beta_2 \eta_0/2)}{1 + \beta_2 \eta_0 \eta_1}
\end{align*}
\]

and for \( \delta_1 = \delta_0 + 2\pi \),

\[
\begin{align*}
x &= \frac{-2\pi \eta_0 (1-f_a) + i_{DC}(1 + \beta_1 \eta_1/2)}{1 + \beta_1 \eta_0 \eta_1}, \\
y &= \frac{2\pi \eta_1 (1-f_a) + i_{DC}(1 + \beta_2 \eta_0/2)}{1 + \beta_2 \eta_0 \eta_1}
\end{align*}
\]

3.3. Equal junctions

We are now ready to discuss the trajectories of the phases \( \delta_0 \) and \( \delta_1 \) on the potential surface, considering first equal junctions, so that \( \eta_0 = \eta_1 = \frac{1}{2} \).

3.3.1. \( f_a = 0 \)

The simplest case is that for \( f_a = 0 \), with a trajectory corresponding to the \( n = 1 \) step shown in Fig. 3(a). Here, the trajectory is along the diagonal \( \delta_1 = \delta_0 \), as it is for all values of \( i_{DC} \), \( i_{AC} \), and step number. Since the potential surface is symmetric about this line due to the trough of the parabolic component, the trajectory simply follows this diagonal. This situation is equivalent to that of a single junction since both phases advance simultaneously, and thus the integral steps have RSJ behavior and there are no half-integral steps, as in Fig. 2(a).
potential for a single junction is the same as that of
the potential surface along the diagonal \( \delta_1 = \delta_0 \) by
setting \( \delta = (\delta_0 + \delta_1)/2 \). For step number \( n \), in one
AC cycle both \( \delta_0 \) and \( \delta_1 \) advance by \( 2\pi n \), or
equivalently the trajectory advances by \( n \) minima.
The total voltage \( V \) across the entire DC SQUID is
given by

\[
V = \frac{1}{2} \sum_{j=0}^{1} \left( V_j + L_j \frac{dL_j}{d\tau} \right).
\]

Converting to dimensionless units and summing the
normalized currents gives a normalized total voltage \( \nu \) of

\[
\nu = \frac{1}{2} \sum_{j=0}^{1} \frac{d\delta_j}{d\tau} + \frac{\beta_L}{4} i_{AC} \Omega \cos(\Omega \tau).
\]

The normalized total DC voltage \( \nu_{DC} \) obtained by
averaging over one or more complete AC cycles is thus

\[
\nu_{DC} = \frac{1}{2} \sum_{j=0}^{1} d\delta_j.
\]

Over one AC cycle, \( d\tau = 2\pi/\Omega \) and \( d\delta = 2\pi n \), so
\( \nu_{DC} = n\Omega \) or, restoring units, \( V_{DC} = (h/2e)n\nu \), the
value given above for Shapiro steps.
The lobes in \( \Delta i_{DC}(i_{AC}) \) shown in Fig. 2(a) corres-
pond to the different number of minima that the
trajectory reaches in one AC cycle [28,29]. Specifi-
cally, for the first lobe for \( n = 0 \) \( (0 \leq i_{AC} \leq 1.2) \), the
trajectory oscillates in the minima labeled by point a
in Fig. 1(a), while for the second lobe \( (1.2 \leq i_{AC} \leq 1.6) \) the
trajectory follows the path a–c–a. Since the
trajectory starts and ends the AC cycle at point a,
there is no DC voltage generated and \( n = 0 \) even
though the trajectory did reach point c. For the first
lobe for \( n = 1 \) \( (0 \leq i_{AC} \leq 1.4) \) the path is a–c, while
for the second lobe \( (1.4 \leq i_{AC} \leq 1.8) \) the path is
a–c–e–c.

3.3.2. \( f_a = 0.5 \)
The much more interesting situation is that of
\( f_a = 0.5 \), for which there are half-integral steps. A
trajectory for the \( n = 1/2 \) step is shown in Fig. 3(b).
The trajectory now follows a zig-zag path on the
potential surface between minima symmetric about
the line \( \delta_1 = \delta_0 + n \), the trough of the parabolic
component of the potential. This zig-zag path is
possible because the minima along the lines \( \delta_1 = \delta_0 \)
and \( \delta_1 = \delta_0 + 2\pi \) are equivalent, as well as being
shifted toward the symmetry line. Using the locations
of the minima from Eqs. (18) to (21) with
\( \beta_L = 10 \) and neglecting the contribution from \( i_{DC} \),
for \( f_a = 0 \) the potential at point a is less than that at
point b by \( 5\pi^2/49 \). However, for \( f_a = 0.5 \) the poten-
tials at points a and b are equal, so the two lines of
minima are equivalent. If \( i_{DC} \) is included, it simply
tilts the potential surface and makes the potential at
point b lower than that at point a by \( \pi i_{DC} \).
The most important conclusion from the case of
\( f_a = 0.5 \) is that for all step numbers \( n \) the trajectory
has a zig-zag path and advances by \( 2n \) minima in
one AC cycle. Thus, the only significant difference
between integral and half-integral steps is that in the
former case the trajectory advances by an even num-
ber of minima, while in the latter case it advances by
an odd number. Consequently, for integral steps the
trajectory begins and ends the cycle at minima along
the same line, so \( \delta_0 \) and \( \delta_1 \) both advance by \( 2\pi n \).
For half-integral steps, however, the trajectory be-
gins a cycle at a minimum along one line and ends
the cycle at a minimum along the other line. There-
fore, \( \delta_0 \) and \( \delta_1 \) alternately advance by either approx-
imately \( 2\pi(n - 1/2) \) or \( 2\pi(n + 1/2) \) in one AC
cycle and both advance by \( 2\pi \times 2n \) in two AC
cycles. More specifically, consider the path b–c–d in
Fig. 1. Using Eqs. (18) to (21), the phase changes in
going from point b to point c are

\[
\Delta \delta_0 = 2\pi - \frac{2\pi \eta_1}{1 + \beta_L \eta_0 \eta_1},
\]

and

\[
\Delta \delta_1 = \frac{2\pi \eta_0}{1 + \beta_L \eta_0 \eta_1},
\]

while from c to d they are

\[
\Delta \delta_0 = \frac{2\pi \eta_1}{1 + \beta_L \eta_0 \eta_1},
\]

and

\[
\Delta \delta_1 = 2\pi - \frac{2\pi \eta_0}{1 + \beta_L \eta_0 \eta_1}.
\]

These phase changes are obviously not 0 or \( 2\pi \),
although both are \( 2\pi \) along the entire path b–c–d.
Thus, averaging over two AC cycles, each phase advances by $2\pi n$ in one cycle, yielding Shapiro steps with voltages for which $n$ is half-integral. This is an extension to two junctions, considering only integral and half-integral Shapiro steps, of the general case for a single junction [26]. There, when the phase $\delta$ advances by $2\pi p$ in $q$ AC cycles, the step number $n = p/q$ results. Also, note especially that a zig-zag trajectory is not the same as $\delta_0$ and $\delta_1$ advancing simultaneously on integral steps and alternately on half-integral steps, as suggested previously [3]. This is further illustrated below.

For there to be a zig-zag trajectory on the potential surface for $0 < f_s < 1$, and thus the possibility of half-integral steps, there must be minima along both the lines $\delta_i = \delta_0$ and $\delta_i = \delta_0 + 2\pi$. Additionally, these minima must be accessible to the trajectory through a combination of the values of the minima, the heights of the barriers between minima, and the forces determining the trajectory. This zig-zag trajectory explains many of the features of $\Delta i_{DC}(i_{AC})$ detailed in Section 3.1. Increasing $f_s$ from 0 to 0.5 causes the minima along the lines $\delta_1 = \delta_0$ and $\delta_1 = \delta_0 + 2\pi$ to approach equivalency, thereby increasing $\Delta i_{DC}$ of the half-integral steps. The value of $\Omega$ for $\Omega < 1$ has virtually no effect on the behavior of $\Delta i_{DC}(i_{AC})$ for either integral or half-integral steps since it determines only the frequency at which the AC current is applied and not the amplitudes of any forces. On the other hand, the value of $\beta_i$ has a significant effect because, from Eq. (13), it determines the steepness of the parabolic component of the potential surface. For $0 < f_s < 0.5$, the steeper this component (the smaller the value of $\beta_i$), the less likely that a minimum along $\delta_i = \delta_0 + 2\pi$ will be present or accessible. Thus, the range of $f_s$ over which half-integral steps occur increases with increasing $\beta_i$. The necessity of including the inductance of the DC SQUID loop is also readily apparent, since in the limit of zero inductance the parabolic component is infinitely steep and the trajectory is confined along the line $\delta_i = \delta_0 + 2\pi f_s$. The lobes of $\Delta i_{DC}(i_{AC})$ are analogous to those for $f_s = 0$, but now the minima along $\delta_i = \delta_0 + 2\pi$ are accessible. For example, for the second lobe of the $n = 0$ step the trajectory follows the path $b-c-b$, while for the third lobe it follows $b-c-d-c-b$. For the $n = \frac{1}{2}$ step, the trajectory for the first lobe follows the path $b-c$ and for the second lobe it follows $b-c-d-c$. Because of this zig-zag trajectory, there are more lobes for $i_{AC} \leq 3.0$ for $f_s = 0.5$ than for $f_s = 0$, in agreement with the results shown in Figs. 2(a) and (b). Also because of this trajectory, the alternation between the maximum values of $\Delta i_{DC}$ with increasing $i_{AC}$ between even and odd integral steps for $f_s = 0$ becomes an alternation between integral and half-integral steps for $f_s = 0.5$. Therefore, the integral steps are coincident with each other and have non-RSJ behavior, and the half-integral steps also have coincident behavior.

3.3.3. Instantaneous junction voltages

A further consequence of the zig-zag trajectory is shown in Fig. 4, where the normalized voltages across the two junctions are plotted as a function of time over two AC cycles for the $n = 0$, $\frac{1}{2}$, and 1.
steps for their first and second lobes in \( \Delta i_{DC}(i_{AC}) \).
From Eq. (3), these voltages are given by Eqs. (8) and (9) after solving for \( \delta_0 \) and \( \delta_1 \) and dividing by the appropriate value of \( \eta_i \). Note that these are the voltages across only the Josephson junction elements; the voltage across the entire DC SQUID is given by Eq. (22). From the relation \( u_j = V_j / L_i R_s = d\delta_j / d\tau \), there is a voltage spike across junction \( j \) whenever the trajectory moves in the \( \delta_j \) direction between the minima. The spike is positive when \( \delta_j \) increases and negative when \( \delta_j \) decreases. The average net number of spikes (obtained by subtracting the number of negative spikes from the number of positive ones) across each junction in two AC cycles is equal to the step number \( n \).

As is readily apparent in Fig. 4, the voltage spikes across the two junctions are never simultaneous with each other because with a zig-zag path the trajectory alternately advances primarily along the \( \delta_0 \) and \( \delta_1 \) directions, but not both simultaneously or one exclusively. For the half-integral steps, the voltage spikes are periodic in two AC cycles. In Fig. 4(c), for the first lobe in \( \Delta i_{DC}(i_{AC}) \) during the first AC cycle the trajectory advances along the path b-c, resulting in a voltage across junction 0, while during the second AC cycle it advances along c-d, with a voltage across junction 1. For the second lobe, in Fig. 4(d), during the first AC cycle the trajectory follows the path b-c-d-c, giving positive voltage spikes across junctions 0 and 1 and a later negative spike across junction 1. The order of the voltage spikes is reversed during the next AC cycle.

The voltages for the integral steps, while periodic in only one AC cycle, also have a complicated behavior, demonstrating again that the two phases do not advance simultaneously for these steps. For the first lobe on the \( n = 0 \) step, in Fig. 4(a), the trajectory remains at point b so there are no voltage spikes. For the second lobe, in Fig. 4(b), the trajectory follows the path b-c-b, so the voltage across junction 0 first has a positive and then a negative spike, while there are no voltage spikes across junction 1. The third lobe has positive and negative spikes across both junctions. With the \( n = 1 \) step, in Figs. 4(e) and 4(f), there are positive voltage spikes across both junctions for the first lobe since the path is b-c-d, while for the second lobe there are a total of three positive and one negative spikes along b-c-d-e-d.

### 3.3.4. Comparisons of static and dynamic results

It is also useful to compare the phase changes from Eqs. (24) and (25), which use the linear approximation of the location of the minima of the potential surface, with phase changes obtained using either numerical calculation of the location of the minima or their actual trajectory from dynamical simulations. The results, using the parameters \( \eta_0 = \eta_1 = \eta_2 = 0.5 \), \( \beta_x = 10 \), \( f_x = 0.5 \), \( i_{DC} = 0.5 \), and \( i_{AC} = 0.31 \), are presented in Table 1. Note in all cases the alternation in the values between \( \delta_0 \) and \( \delta_1 \), which is a consequence of the zig-zag trajectory. The values obtained by the linear approximation agree well with those calculated numerically, thus validating the use of this approximation. Even more important is the agreement between the approximate values calculated from the static positions of the minima of the potential surface and the actual values calculated from the dynamic trajectory. This illustrates the usefulness of the potential-surface description in understanding the dynamics of a DC SQUID.

As an additional example, when the trajectory does not follow the diagonal \( \delta_1 = \delta_0 \), a finite oscillating flux \( f = (\delta_1 - \delta_0)/2\pi \) appears in the loop of the DC SQUID. If the trajectory has a zig-zag pattern, \( f \) oscillates close to, but not at, the values of 0

<table>
<thead>
<tr>
<th>Path</th>
<th>Linear approximation</th>
<th>Numerical calculation</th>
<th>Actual trajectory</th>
</tr>
</thead>
<tbody>
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<td></td>
<td>( \Delta \delta_0/\pi )</td>
<td>( \Delta \delta_1/\pi )</td>
<td>( \Delta \delta_0/\pi )</td>
</tr>
<tr>
<td>b-c</td>
<td>1.714</td>
<td>0.286</td>
<td>1.705</td>
</tr>
<tr>
<td>c-d</td>
<td>0.286</td>
<td>1.714</td>
<td>0.295</td>
</tr>
<tr>
<td>b-c-d</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 1
Normalized changes in phases \( \Delta \delta_0/\pi \) and \( \Delta \delta_1/\pi \) along the indicated paths calculated from the locations of the minima of the potential surface determined using a linear approximation or numerical calculation and from the actual trajectory of the phases.
and 1 since the minima are shifted towards the trough of the parabolic component. Using the same parameters as in the preceding paragraph, the values of $f$ at points b and c are 0.857 and 0.143, 0.854 and 0.146, and 0.841 and 0.159 using the linear approximation, numerical calculation, and actual values from the trajectory, respectively. Again, the approximate values agree well with the actual values.

3.4. Unequal junctions

A more realistic situation for an actual DC SQUID is for the critical currents of the two junctions to be unequal. This leads to more complicated behaviors of $\Delta i_{DC}$, as shown in Figs. 2(c) to (f). The value $\eta_0 = \frac{2}{3}$ was chosen because, while somewhat extreme, it illustrates the primary behaviors and trajectories associated with unequal junctions.

For $f_a = 0$, the trajectory now has a zig-zag path, as shown in Fig. 5(a) for the $n = \frac{1}{2}$ step, instead of a straight line as for equal junctions. The forces from the potential are proportional to Eqs. (14) and (15), and are not equal along the line $\delta_1 = \delta_0$ if $\eta_0 \neq \eta_1$. These unequal forces cause a zig-zag trajectory, and thus half-integral steps are possible for $f_a = 0$. Also, $i_e$ is less than one because the height of the saddle point in the $\delta_1$ direction between minima is less than it is for equal junctions by an amount $\eta_0 - \eta_1$. For $f_a = 0.5$ there is a zig-zag trajectory with pronounced loops near the minima along $\delta_1 = \delta_0 + 2\pi$, as shown in Fig. 5(b) for the $n = \frac{1}{2}$ step. Comparing Figs. 2(b) and (e), $\Delta i_{DC}$ of the half-integral steps decreases

![Fig. 5. Trajectory of $\delta_0$ and $\delta_1$ on the potential surface $u$ for unequal junctions and the parameters $\Omega = 0.175$, $\beta_\iota = 10$, $\eta_0 = \frac{2}{3}$, and $i_{DC} = 0.5$ with the given normalized applied magnetic flux $f_a$ and step number $n$ for (a) $i_{AC} = 0.45$, (b) $i_{AC} = 0.54$, (c) $i_{AC} = 0.79$, and (d) $i_{AC} = 0.33$. The diagonals $\delta_1 = \delta_0$ and $\delta_1 = \delta_0 + 2\pi$, near which minima are located, are indicated by lines.](image-url)
with increasing \( \eta_0 \) since the height of the saddle point in the \( \delta_0 \) direction between minima increases even though the values of the potential at the minima are equal when \( i_{DC} \) is neglected.

The maximum value of \( \Delta i_{DC} \) of the half-integral steps still increases with decreasing \( i_c \) because when \( i_c \) is a minimum a zig-zag trajectory is most likely, and such a trajectory is least likely when \( i_c \) is a maximum. For \( f_s = 0.7 \), \( i_c \) is a maximum from the combination of a relatively large barrier height between points b and c and a relatively small difference between the forces along the primary axes from the potential. The trajectory of the \( n = 1 \) step, shown in Fig. 5(c), is primarily along the diagonal \( \delta_0 \approx \delta_0 + 2\pi \), as expected for \( f_s \) close to one, but there is also some excursion in the troughs between points c and d, etc. Since the trajectory is primarily along a diagonal and does not have a zig-zag path, half-integral steps do not occur. For \( f_s = 0.2 \), \( i_c \) is a minimum, and the usual zig-zag trajectory is present, as shown in Fig. 5(d) for the \( n = \frac{1}{2} \) step. However, from Fig. 2(d), \( \Delta i_{DC} \) for the half-integral steps is relatively large for small \( i_{AC} \) and smaller with a paired and alternating behavior for larger \( i_{AC} \). The maximum value of \( \Delta i_{DC} \) for the half-integral steps decreases with increasing \( \eta_0 \) for the same reason as given for the case \( f_s = 0.5 \).

The paired and alternating behavior of the half-integral steps for \( f_s = 0.2 \) is correlated with a lack of symmetry in the potential surface. Consider the minima at points a, b, and c in Fig. 1, and the barriers between them at the saddle points. Also, neglect the contribution of \( i_{DC} \) to the potential surface, so there is no tilt component, since it only results in a constant force in the diagonal direction. Now, for equal junctions, the barrier heights are the same in either the \( \delta_0 \) or \( \delta_1 \) direction, so there is symmetry about the minima. With \( f_s = 0.5 \), the potential has the same value at points a, b, and c, so there is also symmetry about the saddle points. However, with \( f_s = 0.2 \), the potential at points a and c is less than it is at point b, so this symmetry is not present, although the symmetry about the minima remains. Whenever there is any symmetry, as is always the case for equal junctions, the half-integral steps have coincident behavior.

For unequal junctions, the barrier heights depend on the direction in the \( \delta_0-\delta_1 \) plane, with those in the \( \delta_0 \) direction being larger than those in the \( \delta_1 \) direction, as illustrated in Fig. 1(b). Therefore, there is never any symmetry about the minima. However, for the specific case of \( f_s = 0.5 \), there is symmetry about the saddle points for the same reason as that given above for equal junctions. Therefore, the half-integral steps again have coincident behavior, as in Fig. 2(e). For other values of \( f_s \), such as 0.2, there is no symmetry about either the minima or the saddle points and paired and alternating behavior results, as in Fig. 2(d). There is also coincident behavior for \( f_s = 0 \), as shown in Fig. 2(c), which is likely correlated with a symmetry of the potential surface about the line \( \delta_1 = \delta_0 \). Thus, symmetry considerations provide valuable information about the behaviors of \( \Delta i_{DC}(i_{AC}) \) for half-integral steps.

4. Conclusions

A potential-energy surface with independent phase coordinates \( \delta_0 \) and \( \delta_1 \), was derived for a DC SQUID. This potential surface is composed of a double cosine component resulting from the energy stored in the junctions, a parabolic component with its trough along a diagonal from the magnetic field energy stored in the inductor, and a tilt in the diagonal direction from the applied DC current. This potential surface provides a physical basis for understanding the occurrence and behavior of half-integral Shapiro steps in DC SQUID's and high-\( T_c \) grain-boundary junctions. With this surface, the trajectory can be either nearly straight along one diagonal of minima or have a zig-zag path between minima on two parallel diagonals. For the first kind of trajectory, only integral steps are possible, while both integral and half-integral steps can occur with the second kind. With a zig-zag trajectory, the phases across the junctions change alternately, not simultaneously, which is reflected in the instantaneous voltages across the junctions. For integral steps, the trajectory advances by an even number of minima in one AC cycle, while for half-integral steps it advances by an odd number. Static calculations of the phase changes and flux along a trajectory, based upon the locations of minima derived from a linear approximation of the relevant equations, are in excellent agreement with those values calculated from the actual trajec-
ries. Thus, the minima of the potential surface are primarily responsible for the paths of the trajectories. The different behaviors of $\Delta f_{DC}(i_{AC})$ of both the integral and half-integral steps for equal and unequal junctions are correlated with symmetries of the potential surface.

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