Vibration of Tensioned Beams with Intermediate Damper. 
II: Damper Near a Support

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Abstract: Analytical solutions are used to investigate the free vibrations of tensioned beams with a viscous damper attached transversely near a support. This problem is of particular relevance for stay-cable vibration suppression, but no restrictions on the level of axial load are introduced, and the results are quite broadly applicable. Characteristic equations for both clamped and pinned supports are rearranged into forms suitable for numerical solution by fixed-point iteration, whereby the complex eigenfrequencies and corresponding damping ratios can be accurately computed within a few iterations. Explicit asymptotic approximations for the complex eigenfrequencies are also obtained, subject to restrictions on the closeness of the eigenfrequencies to their undamped values. These asymptotic approximations are expressed in the same “universal” form identified in previous studies. It is observed that the maximum attainable modal damping ratios and the corresponding optimal values of the damper coefficient can be significantly affected by bending stiffness and by the nature of the support conditions, and a nondimensional parameter grouping is identified that enables an assessment of when bending stiffness should be considered.

CE Database subject headings: Vibration; Damping; Eigenvalues; Beams; Cables; Bridges, cable-stayed; Optimization.

Introduction

In the companion paper (Main and Jones 2005), exact analytical solutions were formulated for the complex-valued eigenmodes of tensioned beams with an intermediate damper, and the influence of damper location on the solution characteristics was investigated. Practical constraints commonly limit the attachment point of supplemental dampers to be relatively near a support, and efficient and accurate evaluation of the complex eigenfrequencies in such cases is of significant interest for design purposes, in order to facilitate determination of the appropriate damper sizing to achieve specified...
levels of damping. In this paper, the characteristic equations for both pinned and clamped supports are rearranged into forms that facilitate efficient and accurate numerical solution by fixed-point iteration. These iterative schemes represent generalizations of the iterative scheme previously developed by Krenk (2000) for the taut string, which has also been generalized to the case of a sagging cable by Krenk and Nielsen (2002). Wittrick (1986) previously presented an iterative solution scheme for the undamped eigenfrequencies of tensioned beams with clamped supports, and it is noted that the iterative scheme presented in this paper can also be used to evaluate undamped eigenfrequencies (simply by setting the damper coefficient to zero), while the present scheme has the advantage of remaining applicable for all values of axial load, from the limit of a taut string to the case of a beam without tension. For nonzero values of the damper coefficient, iteration converges rapidly provided the damper remains relatively close to a support; solution characteristics for damper locations further along the span are discussed in Main and Jones (2005).

Asymptotic approximations for the complex eigenfrequencies are also obtained, for more restrictive cases in which the damper is located within a few percent of the span length from a support and the eigenfrequencies are only slightly shifted by the damper. These approximations yield expressions for the optimal damper coefficient, for which the damping ratio in a particular mode is maximized, and enable explicit determination of the damping ratios in each of the first few modes for any given value of the damper coefficient. The asymptotic approximations for both clamped and pinned supports are expressed in the relatively simple “universal” form previously discussed by Krenk and Nielsen (2002). It is noteworthy that this form of approximation remains quite accurate
from the taut-string limit to the case of beam without axial force, provided the real-valued frequency shift produced by fully locking the damper remains sufficiently small. In fact, the applicability of this form of approximation is quite broad, as Main and Krenk (2005) have recently obtained an approximation in this “universal” form for the complex eigenfrequencies of a general discrete system with several viscous dampers, subject to certain restrictions on the closeness of the undamped and fully locked eigensolutions and on the relative sizing of the dampers.

These iterative and asymptotic solutions for tensioned beams are of particular relevance to the design of dampers for stay-cable vibration suppression in bridges. The stays of many cable-stayed bridges have exhibited problematic large-amplitude vibrations, largely as a consequence of their very low levels of inherent mechanical damping (e.g., Yamaguchi and Fujino 1998), and to suppress these vibrations, mechanical dampers are commonly attached to the cables transversely near the anchorages (e.g., Main and Jones 2001). The dynamic characteristics of the resulting cable-damper system have been the subject of much investigation, with the aim of facilitating effective damper design. Because stay cables typically have large levels of axial tension, their dynamic behavior is often accurately modeled as a taut string, and a number of previous studies have used the taut-string approximation to model the dynamics of a cable with attached damper (e.g., Carne 1981, Pacheco et al. 1993, Krenk 2000, Main and Jones 2002). The taut-string approximation neglects the bending stiffness of the cable as well its axial extensibility and sagged equilibrium profile under self-weight.
The influence of sag has been considered in a number of previous studies (e.g., Xu and Yu 1998, Tabatabai and Mehrabi 2000, Krenk and Nielsen 2002), and it has been shown that moderate amounts of sag, characteristic of very long stay cables, can significantly reduce the attainable damping ratios in the first mode, while the damping ratios in higher modes are virtually unaffected. In contrast, relatively few studies have considered the influence of bending stiffness on transverse damping of stay cables. The combined effects of bending stiffness and sag were considered by Tabatabai and Mehrabi (2000), using a finite-difference formulation, and by Christenson (2001), using a Galerkin formulation in the context of an active control study. Both studies showed that the influence of bending stiffness leads to increases in the optimal damper coefficient, with this effect being stronger for damper locations nearer to a support. Although neither study discussed the influence of the support conditions, it is noted Tabatabai and Mehrabi (2000) considered clamped supports, while Christenson (2001) considered pinned supports.

Both clamped and pinned supports are considered in the present study, and importantly, it is found that the differences between the two cases can be significant. When bending effects become important, much larger frequency shifts can be produced by the damper in the case of pinned supports, and therefore, higher levels of damping can be achieved in this case. It is also found that the optimal value of the damper coefficient, for which the damping is maximized in a given mode, can differ significantly between the cases of clamped and pinned supports, being significantly larger in the former case. Importantly for stay-cable vibration suppression, it is found that for realistic combinations of damper location and cable properties, the optimal damper coefficient can
be significantly larger than predicted by the taut-string approximation, which is currently used in design. A nondimensional parameter grouping is identified, applicable for both clamped and pinned supports, which allows for an assessment of whether the influence of bending stiffness should be considered for specified cable properties and damper location.

**Characteristic Equations**

Free lateral vibrations are considered for a uniform axially loaded beam with intermediate viscous damper for both pinned and clamped support conditions, as depicted in Fig. 1. While the problem formulation is presented in detail in Main and Jones (2005), a brief derivation of the characteristic equations, from which the complex eigenfrequencies are determined, is included in the following for completeness. The beam is subjected to constant axial tension $T$ and has mass per unit length $m$ and bending stiffness $EI$. The total span of the beam is $l_0$, and a linear viscous damper with coefficient $c$ is attached at an intermediate point, dividing the beam into two segments of length $l_1$ and $l_2$. With no loading along the span, the equation of motion for each segment is given by the following well-known partial differential equation:

$$EI \frac{\partial^4 y}{\partial x^4_j} - T \frac{\partial^2 y}{\partial x^2_j} + m \frac{\partial^2 y}{\partial t^2} = 0 \quad (1)$$

where $y(x_j, t) =$ transverse deflection and $x_j =$ axial coordinate along the $j^{th}$ segment.

The solution to Eq. (1) is represented in separable form as $y(x_j, t) = Y(x_j)e^{iot}$, where $i = \sqrt{-1}$ and $\omega$ is complex in general, with its real part giving the frequency of damped oscillation and its imaginary part giving the rate of decay. Substituting this form into Eq.
(1) yields a fourth-order ordinary differential equation in the spatial coordinate for the eigenfunction \( Y(x_j) \), and the general solution can be expressed in the following form:

\[
Y(\xi_j) = A_1 e^{-p\xi_j} + A_2 e^{-p(\mu_j-\xi_j)} + A_3 \cos q\xi_j + A_4 \sin q\xi_j
\]  

(2)

where \( \xi_j = x_j/l_0 \) is the nondimensional axial coordinate, \( \mu_j = l_j/l_0 \) is the nondimensional segment length, and \( p \) and \( q \) can be expressed as follows:

\[
pq = \sqrt{\left(\frac{1}{2} \gamma^2 + (\pi\gamma\omega)^2\right)^2 + \frac{1}{2} \gamma^2} = \sqrt{\left(\frac{1}{2} \gamma^2 + (\pi^2 \tilde{\omega})^2\right)^2 + \frac{1}{2} \gamma^2}
\]  

(3)

where \( \gamma = \sqrt{Tl_0^2 / EI} \) is a nondimensional parameter that indicates the relative importance of axial loading and bending stiffness and \( \omega = \omega / \omega_{01}^S \) and \( \tilde{\omega} = \omega / \omega_{01}^B \) are alternative nondimensional frequencies, where \( \omega_{01}^S = \pi / l_0 \sqrt{T / m} \) is the fundamental frequency of a taut string and \( \omega_{01}^B = \pi^2 / l_0^2 \sqrt{EI / m} \) is the fundamental frequency of a beam with pinned supports. Eq. (3) yields the following useful relationships:

\[
p^2 - q^2 = \gamma^2
\]  

(4)

\[
pq = \pi^2 \tilde{\omega} = \pi \gamma \omega
\]  

(5)

When \( \gamma \) is large, the effects of axial tension predominate and the “taut-string” nondimensionalization \( \tilde{\omega} \) is suitable; when \( \gamma \) is small, bending effects predominate and the “beam” nondimensionalization \( \omega \) is suitable. Tabatabai and Mehrabi (2000) report from a database of stay-cable properties that nearly all of the cables had values of \( \gamma \) between 10 and 600, with 82 % of the cables having \( \gamma > 100 \).
For the pinned supports of Fig. 1(a), enforcing boundary conditions and requiring equilibrium of forces and continuity of displacement and slope at the damper location with the spatial form of solution in Eq. (2) leads to the following characteristic equation:

\[ Q^{pp}_0 + i\tilde{c} \frac{Q^{pp}_1 Q^{pp}_2 + Q^{cp}_1 Q^{pp}_2}{2(p^2 + q^2)} = 0 \]  

(6)

in which \( \tilde{c} = c_0 / \sqrt{EI} \) is a “beam” nondimensional damper coefficient, suitable for small values of \( \gamma \). The “taut-string” nondimensionalization \( \hat{c} = c / \sqrt{Tm} \) is suitable for large values of \( \gamma \), where \( \hat{c} = \gamma \hat{c} \). The factors \( Q^{pp}_j \) and \( Q^{cp}_j \) are defined as follows:

\[ Q^{pp}_j = (1 - \varepsilon_j^2) \sin q\mu_j \]  

(7)

\[ Q^{cp}_j = p(1 + \varepsilon_j^2) \sin q\mu_j - q(1 - \varepsilon_j^2) \cos q\mu_j \]  

(8)

where \( \varepsilon_j = \exp(-p\mu_j) \). The superscript \( PP \) in Eq. (7) denotes pinned-pinned supports, and the zeros of \( Q^{pp}_j \) correspond to the eigenfrequencies of a segment of nondimensional length \( \mu_j \) with both ends pinned. Similarly, the superscript \( CP \) in Eq. (8) denotes clamped-pinned supports, and the zeros of \( Q^{cp}_j \) correspond to the eigenfrequencies of a segment of nondimensional length \( \mu_j \) with one end clamped and one end pinned. The subscript \( j = 0 \) in Eq. (6) denotes the total span, with nondimensional length \( \mu_0 = 1 \).

For the clamped supports of Fig. 1(b), enforcing boundary conditions and requiring equilibrium of forces and continuity of displacement and slope at the damper location with the spatial form of solution in Eq. (2) leads to the following characteristic equation:

\[ Q^{cc}_0 + i\tilde{c} \frac{Q^{cc}_1 Q^{cc}_2 + Q^{cp}_1 Q^{cc}_2}{2(p^2 + q^2)} = 0 \]  

(9)
where $Q_{j}^{CP}$ is defined in Eq. (8), and $Q_{j}^{CC}$ can be expressed as a product of two factors:

$$Q_{j}^{CC} = Q_{j}^{CC,S} Q_{j}^{CC,A}$$  \hspace{1cm} (10)

where $Q_{j}^{CC,S}$ and $Q_{j}^{CC,A}$ are defined as follows:

$$Q_{j}^{CC,S} = p(1 - \varepsilon_{j}) \cos \frac{1}{2} q \mu_{j} + q(1 + \varepsilon_{j}) \sin \frac{1}{2} q \mu_{j}$$  \hspace{1cm} (11)

$$Q_{j}^{CC,A} = p(1 + \varepsilon_{j}) \sin \frac{1}{2} q \mu_{j} - q(1 - \varepsilon_{j}) \cos \frac{1}{2} q \mu_{j}$$  \hspace{1cm} (12)

The superscript $CC$ denotes clamped-clamped supports, and the zeros of $Q_{j}^{CC}$ correspond to the eigenfrequencies of a segment of nondimensional length $\mu_{j}$ with both ends clamped. The zeros of $Q_{j}^{CC,S}$ correspond to symmetric modes, while the zeros of $Q_{j}^{CC,A}$ correspond to anti-symmetric modes.

**Solution Features for Damper Near a Support**

It is noteworthy that the characteristic equations in the cases of pinned and clamped supports, Eqs. (6) and (9), have exactly the same form, with the only difference being that each occurrence of $Q_{j}^{PP}$ in Eq. (6) is replaced by $Q_{j}^{CC}$ in Eq. (9). In both cases, the zeros of the first term correspond to undamped solutions ($c = 0$), while the zeros of the quotient in the second term correspond to fully locked solutions ($c \to \infty$), in which the damper acts as an intermediate pin support. As the damper coefficient is increased from zero to approach infinity, the complex eigenfrequencies trace loci in the complex plane. Provided the damper remains relatively near a support, the locus of the $n$th eigenfrequency $\omega_{n}$ exhibits a relatively simple form, originating at the $n$th undamped frequency, denoted $\omega_{0n}$, and terminating at the $n$th fully locked frequency, denoted $\omega_{\infty n}$.
These features are illustrated in Fig. 2(a), which shows the first-mode eigenfrequency locus for the case of clamped supports with $\gamma = 100$ and a damper location of $\mu_i = 0.1$. Arrows indicate the direction of increasing damper coefficient, and the complex mode shapes associated with the points labeled in Fig. 2(a) are plotted in Fig. 2(b). Point $i$ corresponds to the undamped limit ($c = 0$) while point $iii$ is very near the fully locked limit ($c \to \infty$), and point $ii$ corresponds to the case of optimal damping, at which $\text{Im}[\omega]$ is maximized. As discussed in Main and Jones (2005), the eigenfrequency locus of a given mode exhibits significantly different characteristics for cases in which the damper is located near or beyond the first anti-node of the undamped mode shape. However, the present paper focuses on evaluation of the complex eigenfrequencies for cases in which the damper is located relatively near a support and the eigenfrequency loci exhibit the relatively simple form illustrated in Fig. 2(a).

The mode shapes presented in Fig. 2(b) are obtained using nondimensional shape functions derived in the Main and Jones (2005), and in this case of clamped supports, the mode shapes are expressed in terms of the displacement $\alpha_c$ and slope $\theta_c$ at the damper location. The mode shapes are scaled so that $\alpha_c$ is purely real, where the time-varying displacement at the damper is given by $\alpha_c e^{i\omega t}$. With this scaling, the imaginary part of the mode shape is very nearly in phase with the damper force for cases in which the eigenfrequencies are “mostly” real ($\text{Im}[\omega] \ll \text{Re}[\omega]$), since the damper force is proportional to the velocity at the damper, which is given by $i\omega \alpha_c e^{i\omega t}$. This scaling facilitates physical interpretation, because the real part of the mode shape then corresponds nearly to the displaced profile at instants when the damper force is zero,
while the imaginary part corresponds nearly to the displaced profile at instants when the
damper force reaches its peak value. In cases when $\text{Im}[\omega]$ becomes large, the phase
difference between the displacement $\alpha e^{i\omega t}$ and the velocity $i\omega\alpha e^{i\omega t}$ at the damper can
differ significantly from a quarter period, and this physical interpretation is no longer
valid. However, the condition $\text{Im}[\omega] \ll \text{Re}[\omega]$ is satisfied for the loci in Fig. 2(a) and in
many cases of practical interest.

With $\alpha_c$ purely real, the undamped mode shape corresponding to point $i$ in Fig.
2(a) is purely real valued, as shown in Fig. 2(b). Conversely, the mode shape associated
with point $iii$, which is very near the fully locked limit ($c \to \infty$), is almost purely
imaginary, indicating that the peak displacements occur at the same instants as peaks in
the damper force. Interestingly, for finite values of the damper coefficient $c$ with real $\alpha_c$,
it is found that the real part of the mode shape, giving the displaced profile at instants of
zero damper force, resembles the undamped mode shape, while the imaginary part,
giving the displaced profile at instants of peak damper force, resembles the fully locked
mode shape. This is illustrated by the mode shape associated with point $ii$ in Fig. 2, which
corresponds to the case of optimal damping. In this case of optimal damping, the real and
imaginary parts of the complex mode shape have comparable magnitude. As $c$ increases,
the relative magnitude of the imaginary part increases, while that of the real part
decreases. The observation illustrated here, that a complex mode shape can be well
represented as a combination of undamped and fully locked mode shapes, was used by
Main and Krenk (2005) in formulating an approximate solution for the complex
eigenfrequencies of a general discrete system with several viscous dampers.
Iterative Solution for Complex Eigenfrequencies

In the case of pinned supports, the undamped eigenfrequencies are associated with the zeros of \( Q_0^{pp} \), which results from Eq. (6) with \( c = 0 \), and where \( Q_0^{pp} \) is given by Eq. (7) with \( \mu_0 = 1 \). The characteristic equation in this well-known limiting case can then be expressed simply as \( \sin q = 0 \), and its solutions are given by:

\[
q_{0n} = n\pi; \quad p_{0n} = \sqrt{\gamma^2 + (n\pi)^2}
\]

where the subscript 0 denotes the undamped system (\( c = 0 \)), \( n = 1, 2, \ldots \) is the mode number, and the expression for \( p_{0n} \) in Eq. (13) follows from Eq. (4). Using Eq. (5), the undamped eigenfrequencies associated with the solutions in Eq. (13) can be expressed in the alternative nondimensional forms as follows:

\[
\tilde{\omega}_{0n} = n^2 \sqrt{1 + \gamma^2 / (\pi n)^2}; \quad \tilde{\varphi}_{0n} = n\sqrt{1 + (\pi n)^2 / \gamma^2}
\]

(14)

To enable iterative solution for the damped eigenfrequencies even for large values of \( c \), a factor of \( \sin q \) can be identified in the second term of Eq. (6) as well. This is accomplished by noting that \( q\mu_2 = q - q\mu_1 \) and using the well-known trigonometric identities to expand \( \sin q\mu_2 \) and \( \cos q\mu_2 \), which appear in \( Q_2^{pp} \) and \( Q_2^{cp} \). Rearranging the resulting equation to solve for \( \sin q \) and dividing by \( \cos q \) then yields the following equation:

\[
\tan q = \frac{i\tilde{c}N^{pp}}{1 + i\tilde{c}D^{pp}}
\]

(15)

where \( N^{pp} \) and \( D^{cc} \) are defined as follows:

\[
N^{pp} = \frac{P}{P^2 + q^2} \sin^2 q\mu_1
\]

(16)
\[ D_{CC} = \frac{p}{2(p^2 + q^2)} \left[ \sin(2q\mu_n) - \frac{q}{p} \left( \frac{1-e^{-2p\mu_1}}{1-e^{-2p\mu_2}} \right) \right] \] 

(17)

Eq. (15) is well suited for iterative solution by substituting the current estimate for \( q_n \) into the right-hand side and inverting the tangent function to obtain an improved estimate, as follows:

\[ \left[ q_n \right]_{k+1} = n\pi + \tan^{-1}\left( \frac{i\tilde{c}N_{pp}^{PP}}{1 + i\tilde{c}D_{pp}^{PP}} \right) \] 

(18)

where \( n = 1, 2, 3, \ldots \) denotes the mode number and \( \tan^{-1}(\cdot) \) denotes the principal value of the inverse tangent function (i.e., the branch between \(-\pi/2\) and \(\pi/2\) for real-valued arguments). The subscript \( k + 1 \) on the left-hand side of Eq. (18) denotes the improved estimate of \( q_n \), and the subscript \( k \) on the right-hand side indicates that these terms are evaluated using the current estimate. The terms \( N_{pp}^{PP} \) and \( D_{pp}^{PP} \) depend on \( p_n \) as well as \( q_n \), and it is noted that \( p_n \) can be expressed in terms of \( q_n \) using Eq. (4) as 

\[ p_n = (y^2 + q_n^2)^{1/2}. \]

The nondimensional eigenfrequency then follows from \( p_n \) and \( q_n \) using Eq. (5). The undamped values \( q_{0n} \) and \( p_{0n} \), given in Eq. (13), can be used as the initial estimates in the iteration represented by Eq. (18), and the iteration converges rapidly, provided the damped solution remains sufficiently close to the undamped solution that the range of the inverse tangent function is not exceeded.

In the case of clamped supports, the undamped eigenfrequencies are associated with the zeros of \( Q_o^{CC} \), which results from Eq. (9) with \( \tilde{c} = 0 \). According to Eq. (10), \( Q_o^{CC} \) is expressed as a product of two distinct factors, \( Q_o^{CC,S} \) and \( Q_o^{CC,A} \), which correspond to symmetric and anti-symmetric modes and are defined by Eqs. (11) and (12) with \( \mu_0 = 1 \).
To enable iterative solution over the full range of $\gamma$, separate iterative solution schemes are obtained for the odd-numbered and even-numbered damped modes. To solve for the odd-numbered damped modes, the factor associated with symmetric undamped modes is isolated in the first term of Eq. (9) through division by $p(1 - \varepsilon_0)Q_0^{CC,A} \sin(q / 2)$. The first term of Eq. (9) then simplifies to $\cot \frac{1}{2} q + (q / p) \coth \frac{1}{2} p$, and this same quantity can be identified in the second term of Eq. (9) by expanding trigonometric functions and regrouping, as discussed in the Appendix. Rearranging the resulting equation to solve for this quantity yields the following equation:

$$\cot \frac{1}{2} q + \frac{q}{p} \coth \frac{1}{2} p = -\frac{i\varepsilon N^{CC,S}}{1 + i\varepsilon D^{CC,S}}$$

where the rather lengthy expressions for $N^{CC,S}$ and $D^{CC,S}$ are given in Eqs. (49) and (46) in the Appendix. This equation can be rearranged for iterative solution by inverting the cotangent function and using the identities $\cot^{-1} \alpha = \pi / 2 - \tan^{-1} \alpha$ and $\tan^{-1}(-\alpha) = -\tan^{-1} \alpha$:

$$\left[q_n\right]_{k+1} = n\pi + 2\tan^{-1}\left(\frac{q_n}{p_n} \coth \frac{1}{2} p_n + \frac{i\varepsilon N_n^{CC,S}}{1 + i\varepsilon D_n^{CC,S}}\right)$$

where $n = 1, 3, 5, \ldots$ for these odd-numbered modes. No closed-form expression is available for the undamped values $q_{0n}$ and $p_{0n}$ in the case of clamped supports, but the values for pinned supports in Eq. (13), can be used as the initial estimates in the iteration. The iteration converges rapidly, provided the damped solution remains sufficiently close to the undamped solution that the range of the inverse tangent function is not exceeded.

To solve for the even-numbered damped modes in the case of clamped supports, the factor associated with anti-symmetric undamped modes is isolated in the first term of
Eq. (9) through division by \( p(1 + \epsilon_0)Q_0^{C,A} \cos(q/2) \). The first term of Eq. (9) then simplifies to \( \tan \frac{1}{2} q - \frac{q}{p} \tanh \frac{1}{2} p \), and this same quantity can be identified in the second term of Eq. (9) by expanding trigonometric functions and regrouping, as discussed in the Appendix. Rearranging the resulting equation to solve for this quantity yields the following equation:

\[
\tan \frac{1}{2} q - \frac{q}{p} \tanh \frac{1}{2} p = \frac{i\tilde{c}N^{C,A}}{1 + i\tilde{c}D^{C,A}}
\]  

(21)

where \( N^{C,A} \) and \( D^{C,A} \) are defined in Eqs. (57) and (54) in the Appendix. This equation can be rearranged for iterative solution as follows:

\[
[q_n]_{k+1} = n\pi + 2\tan^{-1}\left( \frac{q_n - \tanh \frac{1}{2} p_n + \frac{i\tilde{c}N^{C,A}}{1 + i\tilde{c}D^{C,A}}}{p_n} \right)
\]  

(22)

where \( n = 2, 4, 6, \ldots \) for these even-numbered modes, and the iteration proceeds as described previously for the odd-numbered modes. It is noted the undamped solutions in the case of clamped supports can be efficiently evaluated from Eqs. (20) and (22) with \( \tilde{c} = 0 \), for which second term vanishes in the argument of each inverse tangent function. Unlike the iterative scheme previously presented by Wittrick (1986) for the eigenfrequencies of tensioned beams with clamped supports, the present scheme remains applicable for \( \gamma \to 0 \) as well as for \( \gamma \to \infty \).

**Asymptotic Approximations**

For cases in which the \( n \)th fully locked eigenfrequency \( \omega_{\infty n} \) remains fairly close to the \( n \)th undamped eigenfrequency \( \omega_{0n} \), it is convenient to follow Krenk and Nielsen (2002)
in introducing the notation $\Delta \omega_n$ to represent the complex-valued increment in the $n$th eigenfrequency produced by the damper:

$$
\Delta \omega_n = \omega_n - \omega_{0n}; \quad \Delta \omega_{cn} = \omega_{cn} - \omega_{0n}
$$

(23)

where $\Delta \omega_{cn}$ denotes the real-valued frequency increment produced by fully locking the damper ($c \to \infty$). Explicit asymptotic approximations for $\Delta \omega_n$ can be obtained from the rearranged forms of the characteristic equations in Eqs. (15), (19), and (21) by linearizing the left-hand sides about the undamped eigenfrequency $\omega_{0n}$ and evaluating the right-hand sides at the undamped eigenfrequency $\omega_{0n}$. In each case, these asymptotic approximations can be expressed in the “universal form” discussed by Krenk and Nielsen (2002) as follows:

$$
\Delta \omega_n = \Delta \omega_{cn} \frac{ic / c_n^{opt}}{1 + ic / c_n^{opt}}
$$

(24)

which can be separated into real and imaginary parts as follows:

$$
\text{Re}[\Delta \omega_n] = \Delta \omega_{cn} \frac{(c / c_n^{opt})^2}{1+(c / c_n^{opt})^2}; \quad \text{Im}[\Delta \omega_n] = \Delta \omega_{cn} \frac{c / c_n^{opt}}{1+(c / c_n^{opt})^2}
$$

(25)

As discussed by Krenk and Nielsen (2002), the form of these relations implies that the complex frequency increment $\Delta \omega_n$ traces a semi-circle in the complex plane with diameter $\Delta \omega_{cn}$. The real part $\text{Re}[\Delta \omega_n]$ gives the shift in frequency of damped oscillation, which increases to approach $\Delta \omega_{cn}$ as $c \to \infty$, and the imaginary part $\text{Im}[\Delta \omega_n]$ gives the decay rate, which attains its maximum value, $\text{Im}[\Delta \omega_n^{opt}] = \Delta \omega_{cn} / 2$, when $c = c_n^{opt}$, and decreases to zero as $c \to \infty$. The asymptotic approximations in the
general form of Eq. (24) are found to be quite accurate even in the limit as $c \to \infty$, provided $\Delta \omega_{\infty}$ remains sufficiently small.

In the case of pinned supports, the left hand side of Eq. (15) can be linearized about the undamped solution $q_{0n} = n\pi$ as $\tan q_n = \Delta q_n$, where $\Delta q_n = q_n - q_{0n}$ denotes the complex-valued increment in $q_n$ from the undamped value. In linearizing $\tan q_n$, it is assumed that $|\Delta q_n|$ is fairly small, and Eq. (15) can then be simplified to the following explicit form of approximation by evaluating the right-hand side using the undamped solution:

$$\Delta q_n = \frac{i\tilde{\gamma} N_{0n}^{pp}}{1 + i\tilde{\gamma} D_{0n}^{pp}}$$ \hspace{1cm} (26)

where the subscript $0n$ on the right-hand side indicates that these terms are evaluated using the undamped values of $q_{0n}$ and $p_{0n}$ given in Eq. (13). Using Eqs. (4) and (5), the “beam” nondimensional frequency $\tilde{\omega}_n$ can be expressed in terms of $q_n$ as

$$\tilde{\omega}_n = q_n (\gamma^2 + q_n^2)^{1/2} / \pi^2.$$ While this relationship is nonlinear, it can be linearized about the undamped value $q_{0n}$, assuming again that $|\Delta q_n|$ is fairly small, to give the following approximation:

$$\Delta \tilde{\omega}_n = \frac{p_{0n}^2 + q_{0n}^2}{\pi^2 p_n^0} \Delta q_n$$ \hspace{1cm} (27)

Eq. (27) applies for both pinned and clamped supports, where $q_{0n}$ and $p_{0n}$ are given by Eq. (13) in the case of pinned supports but must be evaluated numerically in the case of clamped supports. Combining Eqs. (26) and (27), an asymptotic approximation for the
complex frequency increment $\Delta \tilde{\omega}_n$ can then be obtained in the “universal form” of Eq. (24), where the limiting real-valued frequency increment $\Delta \tilde{\omega}_{n}^{\infty}$ is approximated as

$$\Delta \tilde{\omega}_{n}^{\infty} = \frac{p_{nn}^2 + q_{nn}^2}{\pi^2 p_n^0} \frac{N_{nn}^{pp}}{D_{nn}^{pp}}$$

and the optimal damper coefficient $c_{n}^{\text{opt}}$ is approximated as

$$c_{n}^{\text{opt}} = 1 / D_{nn}^{pp}$$

While the asymptotic approximation in Eq. (24) is expressed in terms of dimensional quantities, it is noted that either the “beam” or the “taut-string” nondimensionalizations can be used, provided they are used consistently. The “beam” nondimensionalizations of Eqs. (28) and (29) can be readily expressed in terms of the “taut-string” nondimensionalizations by noting that $\Delta \tilde{\omega}_{n}^{\infty} = (\pi / \gamma) \Delta \tilde{\omega}_{n}^{\infty}$ and $\tilde{c} = \tilde{c} / \gamma$.

In the case of clamped supports, it is found that asymptotic approximations for the complex frequency increment $\Delta \tilde{\omega}_n$ can be obtained with the same “universal form” of Eq. (24). However, the expressions for $\Delta \tilde{\omega}_{n}^{\infty}$ and $c_{n}^{\text{opt}}$ with clamped supports differ from those in Eqs. (28) and (29), and distinct expressions are obtained for the odd-numbered and even-numbered modes. For the odd-numbered modes, the left-hand side of Eq. (19) can be linearized about undamped solutions as

$$\frac{\cot(q / 2) + (q / p) \coth(p / 2)}{2 p_{nn}^2} = -\frac{\gamma^2}{2 p_{nn}^2} \left[ 1 + \frac{\cot(\frac{1}{2} q_{nn})}{\frac{1}{2} q_{nn}} + \cot^2(\frac{1}{2} q_{nn}) \right]$$

The latter expression in Eq. (30) has been simplified using trigonometric identities and the relation $\cot(\frac{1}{2} q_{nn}) = -(q_{nn} / p_{nn}) \coth(\frac{1}{2} p_{nn})$, which is satisfied for symmetric
undamped solutions, according to Eq. (19) with $\tilde{c} = 0$. With the left-hand side linearized and the right-hand side evaluated at the undamped solution, Eq. (19) can then be approximated for small $|\Delta q_n|$ as

$$\Delta q_n = \frac{i\tilde{c}N_{0n}^{CC,S}}{L_{0n}^{CC,S}} \frac{1 + i\tilde{c}D_{0n}^{CC,S}}{1}$$

where $n = 1, 3, 5, \ldots$ for these odd-numbered modes, and the subscript $0n$ indicates that these terms are evaluated using the undamped values of $q_{0n}$ and $p_{0n}$. Using Eq. (27) to linearize the complex frequency increment $\Delta \tilde{\omega}_n$ about an undamped solution, an asymptotic approximation for $\Delta \tilde{\omega}_n$ can then be obtained in the “universal form” of Eq. (24), where $\Delta \tilde{\omega}_{\alpha,a}$ and $c_{n}^{\text{opt}}$ in this case are approximated as

$$\Delta \tilde{\omega}_{\alpha,a} = \frac{p_{0n}^2 + q_{0n}^2}{\pi^2 p_n} \frac{N_{0n}^{CC,S}}{D_{0n}^{CC,S} L_{0n}^{CC,S}}$$

$$c_{n}^{\text{opt}} = 1 / D_{0n}^{CC,S}$$

Eqs. (32) and (33) depend on the undamped values of $q_{0n}$ and $p_{0n}$, and for maximum accuracy, these values can be computed iteratively from Eq. (20) with $\tilde{c} = 0$. However, for moderately large values of $\gamma$, the nature of the supports has a relatively small influence on these undamped values, and this influence decreases in the higher modes. In such cases, relatively small error is introduced by simply using the values in Eq. (13) for pinned supports, yielding a fully explicit approximation for the complex eigenfrequencies.

For the even-numbered modes, the left-hand side of Eq. (21) can be linearized about undamped solutions as $\tan(q / 2) + (q / p) \tanh(p / 2) = L_{0n}^{CC,A} \Delta q_n$, where
The latter expression in Eq. (34) has been simplified using trigonometric identities and the relation \( \tan(\frac{1}{2} q_{on}) = \frac{q_{on}}{p_{on}} \tanh(\frac{1}{2} p_{on}) \), which is satisfied for anti-symmetric undamped solutions, according to Eq. (21) with \( \tilde{c} = 0 \). With the left-hand side linearized and the right-hand side evaluated at the undamped solution, Eq. (21) can then be approximated for small \( |\Delta q_n| \) as

\[
\Delta q_n = \frac{i\tilde{c}N_{on}^{CC,A}/L_{on}^{CC,A}}{1 + i\tilde{c}D_{on}^{CC,A}} \tag{35}
\]

where \( n = 2, 4, 6, \ldots \) for these even-numbered modes. For maximum accuracy, these undamped values can be computed iteratively from Eq. (22) with \( \tilde{c} = 0 \). Alternatively, for moderately large values of \( \gamma \), the values for pinned supports in Eq. (13) can be used with relatively small error, as noted previously for the odd-numbered modes. Combining Eqs. (35) and (27), an asymptotic approximation for the complex frequency increment \( \Delta \tilde{\omega}_n \) can be obtained in the “universal form” of Eq. (24), where \( \Delta \tilde{\omega}_{xn} \) and \( c_n^{opt} \) in this case are given by

\[
\Delta \tilde{\omega}_{xn} = \frac{P_{on}^2 + q_{on}^2}{\pi^2 p_n^2} \frac{N_{on}^{CC,A}}{D_{on}^{CC,A} f_{on}^{CC,A}} \tag{36}
\]

\[
c_n^{opt} = 1 / D_{on}^{CC,A} \tag{37}
\]

Fig. 3 shows loci of the normalized first-mode eigenfrequency \( \omega / \omega_{q1} \) in the complex plane for a damper location of \( \mu_x = 0.02 \) and for several different values of \( \gamma \). Fig. 3(a) presents results for pinned supports and Fig. 3(b) for clamped supports. In both
graphs, discrete symbols represent exact values computed iteratively, while solid lines represent the asymptotic approximations in the form of Eq. (24). In the case of clamped supports, good agreement is observed between the exact and asymptotic solutions for all values of $\gamma$. It is noteworthy that the semi-circular asymptotic form of the eigenfrequency loci persists over this entire range, from the limit of a taut string to that of a beam without axial tension. In fact, the applicability of the relatively simple form of approximation in Eq. (24) is quite broad, as Main and Krenk (2005) have recently extended this form of approximation to a general discrete system with several viscous dampers, subject to certain restrictions on the closeness of the undamped and fully locked solutions and, in the case of multiple dampers, on the relative sizing of the dampers.

In the case of pinned supports, good agreement is observed for $\gamma=1000$ and $\gamma=50$, but discrepancies become more significant for $\gamma=10$ and $\gamma=0$. For these smaller values of $\gamma$, the difference $\Delta \omega_{\infty}$ between the undamped and fully locked eigenfrequencies becomes fairly large ($\Delta \omega_{\infty} / \omega_{\infty} = 0.61$ for $\gamma=0$), and the assumption of small $|\Delta q_\omega|$, which was used in obtaining the asymptotic approximations, becomes less appropriate. It is noted that fully locking the damper near a pinned support essentially corresponds to changing the support condition from pinned to clamped, and when bending effects become important, this produces much larger frequency shifts than fully locking the damper near a clamped support. These larger frequency shifts are associated with larger attainable values of the decay rate $\text{Im}[\omega_\gamma]$, corresponding to higher damping, and it is clear in Fig. 3 that when $\gamma$ becomes small, much higher damping can be achieved with pinned supports than with clamped supports.
Applications to Stay-Cable Damper Design

A nondimensional design curve has previously been obtained that relates the damping ratios in the first few modes to the viscous damper coefficient in the taut-string limit ($\gamma \to \infty$), for damper locations within a few percent of the span length from a support. This “universal estimation curve” was identified from numerical computation by Pacheco et al. (1993), and Krenk (2000) later obtained an analytical expression for this curve, generalizing an approximation for the first mode previously obtained by Carne (1981). In the present notation, this expression can be written as:

$$\frac{\zeta_n}{\mu_i} = \frac{\pi \hat{c}_n \mu_i}{1 + (\pi \hat{c}_n \mu_i)^2}$$

(38)

where the damping ratio is defined as $\zeta_n = \text{Im}[\omega_n]/|\omega_n|$, and it is noted that Eq. (38) corresponds to the imaginary part of the “universal form” of approximation in Eq. (25). Eq. (38) predicts a maximum attainable damping ratio in mode $n$ of $\zeta_n^{opt}/\mu_i = 1/2$, associated with a corresponding optimal damper tuning given by $\hat{c}_n^{opt} n\mu_i = 1/\pi$. This “universal estimation curve” is of great utility in the design of dampers for cable vibration suppression, and the influence of bending stiffness on this relation and the associated optimal values is therefore of significant practical interest.

Fig. 4 shows graphs of $\zeta_n/\mu_i$ against $\hat{c}n\mu_i$ in the first mode for both pinned and clamped supports, with $\mu_i = 0.02$ and for three different values of $\gamma$. As in Fig. 3, discrete symbols represent exact values computed iteratively, while solid lines represent the asymptotic approximations in the form of Eq. (24), and good agreement is observed between the exact and asymptotic solutions for all values of $\gamma$ in these plots. While only values corresponding to the first mode are plotted in Fig. 4, the mode number $n$ is
retained in the ordinate and abscissa because curves corresponding to the first few modes are observed to coincide very nearly with those shown for the first mode. In the case of $\gamma = 1000$, the nature of the support conditions has a fairly insignificant effect, and the corresponding curves in Figs. 4(a) and 4(b) are essentially equivalent, agreeing quite closely with the taut-string approximation of Eq. (38), which is not shown. However, the evolution of these curves with increasing bending stiffness (decreasing $\gamma$) depends strongly on the nature of the support conditions. Decreasing $\gamma$ leads to significant increases in the maximum attainable damping ratio in the case of pinned supports, but leads to small decreases in the maximum attainable damping ratio in the case of clamped supports. For both types of support conditions, decreasing $\gamma$ leads to increases in the optimal damper coefficient, but this increase is much stronger in the case of clamped supports. Designing a damper using Eq. (38) can thus lead to significantly lower damping than expected in the case of clamped supports, as can be seen in Fig. 4(a), where the curves for $\gamma = 100$ and $\gamma = 50$ lie significantly below the optimal portion of the curve for $\gamma = 1000$.

Fig. 5 shows numerically computed values of the normalized optimal damper coefficient $\hat{c}_{n}^{\text{opt}} n \mu_i$ in the first mode plotted against the product $\gamma \mu_i$ for both pinned and clamped supports and for several different values of $\mu_i$. Only values of $\gamma > 10$ included in the graphs, where $\gamma = 10$ corresponds to the lower limit for most stay cables as reported by Tabatabai and Mehrabi (2000). With this restriction, it is found that the results corresponding to different values of $\mu_i$ collapse quite well along a single curve. The relevance of the product $\gamma \mu_i$, as seen in Fig. 4, stems largely from the important influence
of terms involving \( \epsilon_i = \exp(-p \mu_i) \), where it follows from Eq. (4) that \( p = \gamma \) when \( \gamma^2 \gg q^2 \), and therefore \( \epsilon_i = \exp(-\gamma \mu_i) \). The restriction \( \gamma > 10 \) on the curves in Fig. 5 insures that the condition \( \gamma^2 \gg q^2 \) is reasonably satisfied. While only values corresponding to the first mode are shown in Fig. 5, the mode number \( n \) is retained in the ordinate because values corresponding to the first few modes are observed to fall very nearly on the same curves as those for the first mode. Fig. 5 shows that the optimal damper coefficient is much more strongly affected by decreasing \( \gamma \mu_i \) in the case of clamped supports than in the case of pinned supports, while dramatic increases can be observed in both cases.

Fig. 6(a) shows numerically computed values of the normalized maximum attainable damping ratio \( \zeta_{n,\text{opt}} / \mu_i \) in the first mode plotted against the product \( \gamma \mu_i \) for the case of pinned supports and for several different values of \( \mu_i \). As in Fig. 5, only values of \( \gamma > 10 \) are included in the plot, and the values corresponding to different damper locations are well consolidated along a single curve. Fig. 6(a) shows that in the case of pinned supports the influence of bending stiffness can significantly increase the maximum attainable damping ratios above their values in the taut-string limit.

The situation is different in the case of clamped supports, and in Fig. 6(b) the numerically computed values of \( \zeta_{n,\text{opt}} / \mu_i \) in the first mode are plotted directly against \( \gamma \), rather than against the product \( \gamma \mu_i \) as in Fig. 6(a). In the case of clamped-clamped supports, the ratio \( \zeta_{n,\text{opt}} / \mu_i \) is seen to be relatively constant with \( \gamma \), and a linear scale is used on the ordinate rather than a logarithmic scale as in Fig. 6(a). For values of \( \gamma \) between 10 and 1000, encompassing virtually all stay cables, the influence of bending
stiffness is seen to produce slight reductions of the maximum attainable damping ratios below the taut-string value of $\zeta_{\text{opt}} / \mu = 1 / 2$. Further reductions of $\gamma$ lead to moderate increases in the maximum attainable damping ratio, tending to a value of about 0.8 as $\gamma \to 0$. These increases are not nearly as strong as those observed in the case of pinned supports in Fig. 6(a), and Fig. 6 clearly shows that when bending effects become important, more damping can be added to a given mode with pinned supports than with clamped supports.

For design purposes, it is useful to identify conditions under which the influence of bending stiffness can be neglected and the simple taut-string approximation of Eq. (38) can be used with acceptable accuracy. The nondimensional parameter grouping $\gamma \mu_i$, whose relevance was illustrated in Figs. 5 and 6, is useful for this purpose. Provided that $\gamma \mu_i > 10$, the results shown in Fig. 5 indicate that the optimal damper coefficient is within 20% of the taut-string value in the case of clamped supports and within 9% of the taut-string value in the case of pinned supports. It should be noted that these results are subject to the further restriction that $\gamma > 10$, which is satisfied for virtually all stay cables. The results shown in Fig. 6(a) indicate that for $\gamma \mu_i > 10$ (with $\gamma > 10$), the maximum damping ratio is within 12% of the taut-string value ($\zeta_{\text{opt}} / \mu = 1 / 2$) in the case of pinned supports. In the case of clamped supports the parameter $\gamma$, rather than $\gamma \mu_i$, was found to consolidate the curves of maximum damping ratios, and the results shown in Fig. 6(b) indicate that the maximum damping ration is within 18% of the taut-string value provided that $\gamma > 10$. 
Based on these observations, the condition $\gamma \mu_i > 10$ (with $\gamma > 10$) is proposed as an approximate bound on the applicability of the taut-string “universal estimation curve” (Eq. (38)), beyond which the effects of bending stiffness and the nature of the support conditions should be taken into consideration. For $\gamma = 600$, which corresponds to the upper limit for most stay cables (Tabatabai and Mehrabi 2000), this condition requires that the damper be placed such that $\mu_i > 1.7\%$. A stiffer stay cable with $\gamma = 100$ would require damper placement such that $\mu_i > 10\%$, which is generally not achievable due to practical constraints. The database of stay-cable properties cited in Tabatabai and Mehrabi (2000) indicates that about 18% of stay cables have $\gamma < 100$, and for such cables, consideration of bending effects may be unavoidable. While only the limiting cases of pinned and clamped supports are considered in this paper, it is noted that the anchorage of a stay cable would generally exhibit finite rotational stiffness, placing it somewhere between these limiting cases. Because such dramatically different results are obtained in the two limiting cases, more detailed analysis of the anchorage zone may be required in order to determine a representative rotational stiffness for the support in cases when bending effects are expected to be important. It may even be desirable to intentionally design the anchorage to permit rotation, since pinned supports have been observed to result in higher attainable damping ratios with lower values of the optimal damper coefficient.

**Conclusions**

An analytical solution has been used to investigate the free vibrations of tensioned beams with a viscous damper attached transversely near a support. Characteristic equations for
the cases of clamped and pinned supports have been rearranged into forms suitable for
solution by fixed-point iteration, whereby the complex eigenfrequencies and
corresponding damping ratios can be accurately computed within a few iterations.
Explicit asymptotic approximations for the complex eigenfrequencies have also been
obtained, for cases in which the damper is located within a few percent of the span length
from the support and the eigenfrequencies are only slightly shifted by the damper. For
both clamped and pinned supports, the asymptotic approximations have been obtained in
the same “universal” form identified in other recent studies of distributed systems with
concentrated damping. This form of approximation implies that the eigenfrequencies
trace semi-circles in the complex plane with increasing damper coefficient, originating at
the real-valued undamped eigenfrequencies and terminating at the slightly higher
frequencies associated with fully locking the damper. While the general form of the
asymptotic approximation is the same for both clamped and pinned supports, it was found
that when bending effects become important, much larger frequency shifts can be
produced by the damper in the case of pinned supports, and therefore, higher levels of
damping can be achieved with pinned supports. The optimal value of the damper
coefficient, for which the largest damping is achieved in a given mode, was also found to
differ significantly for clamped and pinned supports when bending effects become
important, being significantly larger for clamped supports. Importantly for stay-cable
vibration suppression, it was observed that the effects of bending stiffness could be
significant for realistic combinations of damper location and cable properties, and a
nondimensional parameter grouping has been identified which enables an assessment of
whether the effects of bending stiffness can be neglected.
Appendix

In obtaining Eq. (19), used for iterative evaluation of the odd-numbered complex eigenfrequencies in the case of clamped supports, it is necessary to rearrange the second term of Eq. (9) in order to identify a factor of the quantity \( \cot \frac{q}{2} + (q/p) \coth \frac{1}{2} p \), whose solutions are associated with symmetric undamped modes. To accomplish this the quantity \( Q_{2}^{CC,S} \), defined by Eq. (11), can be expressed as follows by expanding the terms \( \sin(\frac{1}{2}q\mu) \) and \( \cos(\frac{1}{2}q\mu) \), noting that \( \mu = 1 - \mu_{1} \), and grouping terms on \( \cos \frac{1}{2} q \) and \( \sin \frac{1}{2} q \):

\[
Q_{2}^{CC,S} = P_{2}^{CC,S} \cos \frac{1}{2} q + R_{2}^{CC,S} \sin \frac{1}{2} q
\] (39)

where \( P_{2}^{CC,S} \) and \( R_{2}^{CC,S} \) are defined as

\[
P_{2}^{CC,S} = p(1 - \varepsilon_{2}) \cos \frac{1}{2} q \mu_{1} - q(1 + \varepsilon_{2}) \sin \frac{1}{2} q \mu_{1}
\] (40)

\[
R_{2}^{CC,S} = p(1 - \varepsilon_{2}) \sin \frac{1}{2} q \mu_{1} + q(1 + \varepsilon_{2}) \cos \frac{1}{2} q \mu_{1}
\] (41)

Similarly, the quantity \( Q_{2}^{CP} \), defined by Eq. (8), can be expressed as follows by expanding the terms \( \sin q \mu_{2} \) and \( \cos q \mu_{2} \), and making the substitution \( \sin q = 2 \sin \frac{1}{2} q \cos \frac{1}{2} q \)

\[
Q_{2}^{CP} = 2P_{2}^{CP} \sin \frac{1}{2} q \cos \frac{1}{2} q - R_{2}^{CP} \cos q
\] (42)

where \( P_{2}^{CP} \) and \( R_{2}^{CP} \) are defined as

\[
P_{2}^{CP} = p(1 + \varepsilon_{2}^{2}) \cos q \mu_{1} - q(1 - \varepsilon_{2}^{2}) \sin q \mu_{1}
\] (43)

\[
R_{2}^{CP} = p(1 + \varepsilon_{2}^{2}) \sin q \mu_{1} + q(1 - \varepsilon_{2}^{2}) \cos q \mu_{1}
\] (44)
The characteristic Eq. (9), having been divided by \( p(1 - \varepsilon_0)Q_0^{CC,A} \sin(q/2) \) to isolate the quantity of interest in the first term, can then be expressed as follows:

\[
\cot \frac{q}{2} + \left( \frac{q}{p} \right) \coth \frac{q}{2} + i \hat{c} \left( D^{CC,S} \cot \frac{q}{2} + M^{CC,S} \right) = 0
\]  

(45)

where \( D^{CC,S} \) and \( M^{CC,S} \) are defined in terms of the quantities in Eqs. (40), (41), (43), and (44), as well as those defined previously in Eqs. (8), (10), and (12):

\[
D^{CC,S} = \frac{Q_1^{CC} P_2^{CC,S} Q_2^{CC,A} + 2Q_1^{CC} P_2^{CC} \sin \frac{q}{2}}{2p(p^2 + q^2)(1 - e^{-p})Q_0^{CC,A}}
\]  

(46)

\[
M^{CC,S} = \frac{Q_1^{CC} R_2^{CC,S} Q_2^{CC,A} - Q_1^{CC} R_2^{CC} \cos q \csc \frac{q}{2}}{2p(p^2 + q^2)(1 - e^{-p})Q_0^{CC,A}}
\]  

(47)

Finally, adding and subtracting \( i \hat{c}D^{CC,S}(q/p) \coth \frac{q}{2} \) in the second term of Eq. (45) allows the quantity \( i \hat{c}D^{CC,S} \) to be grouped with the first term as a factor of \( \cot \frac{q}{2} + \left( \frac{q}{p} \right) \coth \frac{q}{2} \), and the characteristic equation takes the following form:

\[
\left( 1 + i \hat{c}D^{CC,S} \right) \left[ \cot \frac{q}{2} + \left( \frac{q}{p} \right) \coth \frac{q}{2} \right] + i \hat{c}N^{CC,S} = 0
\]  

(48)

where \( N^{CC,S} \) is defined as

\[
N^{CC,S} = M^{CC,S} - D^{CC,S}(q/p) \coth \frac{q}{2}
\]  

(49)

Division of Eq. (48) by \( 1 + i \hat{c}D^{CC,S} \) then leads directly to Eq. (19).

In obtaining Eq. (21), used for iterative evaluation of the even-numbered complex eigenfrequencies, it is necessary to rearrange the second term of the characteristic Eq. (9) to identify a factor of the quantity \( \tan \frac{q}{2} - (q/p) \tanh \frac{q}{2} \), whose solutions are associated with the anti-symmetric undamped modes. To accomplish this the term \( Q_2^{CC,A} \), defined by Eq. (12), can be expressed as follows by expanding the terms \( \sin(q/2 \mu_2) \) and \( \cos(q/2 \mu_2) \), noting that \( \mu_2 = 1 - \mu_1 \), and grouping terms on \( \cos \frac{q}{2} \) and \( \sin \frac{q}{2} \):
\[ Q_{2}^{CC,A} = P_{2}^{CC,A} \sin \frac{1}{2}q - R_{2}^{CC,A} \cos \frac{1}{2}q \]  

(50)

where \( P_{2}^{CC,A} \) and \( R_{2}^{CC,A} \) are defined as

\[ P_{2}^{CC,A} = p(1 + \varepsilon_{2}) \cos \frac{1}{2}q \mu_{1} - q(1 - \varepsilon_{2}) \sin \frac{1}{2}q \mu_{1} \]  

(51)

\[ R_{2}^{CC,A} = p(1 + \varepsilon_{2}) \sin \frac{1}{2}q \mu_{1} + q(1 - \varepsilon_{2}) \cos \frac{1}{2}q \mu_{1} \]  

(52)

The characteristic Eq. (9), having been divided by \( p(1 + \varepsilon_{0})Q_{0}^{CC,A} \cos(q / 2) \) to isolate the quantity of interest in the first term, can then be expressed as follows:

\[ \tan \frac{1}{2}q - (q / p) \tanh \frac{1}{2}p + i \tilde{c} \left(D_{CC,A}^{CC,A} \tan \frac{1}{2}q - M_{CC,A}^{CC,A}\right) = 0 \]  

(53)

Where \( D_{CC,A}^{CC,A} \) and \( M_{CC,A}^{CC,A} \) are defined in terms of the quantities in Eqs. (43), (44), (51), and (52), as well as those defined previously in Eqs. (8), (10), and (11):

\[ D_{CC,A}^{CC,A} = \frac{Q_{1}^{CP}Q_{2}^{CC,S}P_{2}^{CC,A} + 2Q_{1}^{CC}P_{2}^{CP} \cos \frac{1}{2}q}{2p(p^2 + q^2)(1 + e^{-p})Q_{0}^{CC,S}} \]  

(54)

\[ M_{CC,A}^{CC,A} = \frac{Q_{1}^{CP}Q_{2}^{CC,S}R_{2}^{CC,A} + Q_{1}^{CC}R_{2}^{CP} \cos q \sec \frac{1}{2}q}{2p(p^2 + q^2)(1 + e^{-p})Q_{0}^{CC,S}} \]  

(55)

Finally, adding and subtracting \( i \tilde{c}D_{CC,A}^{CC,A} (q / p) \tanh \frac{1}{2}p \) in the second term of Eq. (53) allows the quantity \( i \tilde{c}D_{CC,A}^{CC,A} \) to be grouped with the first term as a factor of \( \tan \frac{1}{2}q - (q / p) \tanh \frac{1}{2}p \), and the characteristic equation takes the following form:

\[ \left(1 + i \tilde{c}D_{CC,A}^{CC,A}\right) \left[\tan \frac{1}{2}q - (q / p) \tanh \frac{1}{2}p\right] - i \tilde{c}N_{CC,A}^{CC,A} = 0 \]  

(56)

where \( N_{CC,A}^{CC,A} \) is defined as

\[ N_{CC,A}^{CC,A} = M_{CC,A}^{CC,A} - D_{CC,A}^{CC,A} (q / p) \tanh \frac{1}{2}p \]  

(57)

Division of Eq. (57) by \( 1 + i \tilde{c}D_{CC,A}^{CC,A} \) then leads directly to Eq. (19).
References


Figures

Fig. 1. Tensioned beams with intermediate damper. (a) Pinned supports; (b) Clamped supports.

Fig. 2. (a) Typical form of eigenfrequency locus as $c$ goes from 0 to $\infty$ (arrows indicate increasing $c$) for damper near support (mode 1, $\mu_i = 0.1$, $\gamma = 100$, clamped supports); (b) Corresponding complex mode shapes: real part — , imaginary part — , damper location ○.
Fig. 3. Locus of first-mode eigenfrequency as $c$ goes from 0 to $\infty$ for different values of $\gamma$ ($\mu_i = 0.02$). (a) Pinned supports; (b) Clamped supports.

Fig. 4. Influence of bending stiffness on “universal estimation curve” ($\mu_i = 0.02, n = 1$). (a) Pinned supports; (b) Clamped supports.
Fig. 5. Influence of bending stiffness on optimal damper coefficient ($\gamma > 10$, $n = 1$). (a) Pinned supports; (b) Clamped supports.

Fig. 6. Influence of bending stiffness on optimal damping ratios ($n = 1$). (a) Pinned supports ($\gamma > 10$); (b) Clamped supports.