Separation of variables in an asymmetric cyclidic coordinate system

H. S. Cohl and H. Volkmer

1Applied and Computational Mathematics Division, National Institute of Standards and Technology, Gaithersburg, Maryland 20899-8910, USA
2Department of Mathematical Sciences, University of Wisconsin–Milwaukee, P. O. Box 413, Milwaukee, Wisconsin 53201, USA

(Received 21 January 2013; accepted 7 June 2013; published online 27 June 2013)

A global analysis is presented of solutions for Laplace’s equation on three-dimensional Euclidean space in one of the most general orthogonal asymmetric confocal cyclidic coordinate systems which admit solutions through separation of variables. We refer to this coordinate system as five-cyclide coordinates since the coordinate surfaces are given by two cyclides of genus zero which represent inversions of each other with respect to the unit sphere, a cyclide of genus one, and two disconnected cyclides of genus zero. This coordinate system is obtained by stereographic projection of sphero-conal coordinates on four-dimensional Euclidean space. The harmonics in this coordinate system are given by products of solutions of second-order Fuchsian ordinary differential equations with five elementary singularities. The Dirichlet problem for the global harmonics in this coordinate system is solved using multiparameter spectral theory in the regions bounded by the asymmetric confocal cyclidic coordinate surfaces. © 2013 AIP Publishing LLC.

I. INTRODUCTION

In 1894, Maxime Bôcher’s book “Über die Reihenentwickelungen der Potentialtheorie” was published. It took its origin from lectures given by Felix Klein in Göttingen (see for instance, Refs. 7 and 8). In Bôcher’s book, the author gives a list of 17 inequivalent coordinate systems in three dimensions in which the Laplace equation admits separated solutions of the form

\[ U(x, y, z) = R(x, y, z)w_1(s_1)w_2(s_2)w_3(s_3), \]

where the modulation factor \( R(x, y, z) \) (see p. 519 of Ref. 11) is a known and fixed function, and \( s_1, s_2, s_3 \) are curvilinear coordinates of \( x, y, z \). The functions \( w_1, w_2, w_3 \) are solutions of second order ordinary differential equations. The symmetry group of Laplace’s equation is the conformal group and equivalence between various separable coordinate systems is established by the existence of a conformal transformation which maps one separable coordinate system to another.

In general, the coordinate surfaces (called confocal cyclides) are given by the zero sets of polynomials in \( x, y, z \) of degree at most four which can be broken up into several different subclasses. For instance, eleven of these coordinate systems have coordinate surfaces which are given by confocal quadrics (Systems 1–11 on p. 164 of Ref. 9), nine are rotationally-invariant (Systems 2, 5–8 on p. 164 and Systems 14–17 on p. 210 of Ref. 9), four are cylindrical (Systems 1–4 of Ref. 9), and the five most general are of the asymmetric type, namely confocal ellipsoidal, paraboloidal, spherocylindrical, and two cyclidic coordinate systems (Systems 9–11 on p. 164 and Systems 12 and 13 on p. 210 of Ref. 9). Bôcher² showed how to solve the Dirichlet problem for harmonic functions on regions bounded by such confocal cyclides. However, it is stated repeatedly in Bôcher’s book that the presentation lacked convergence proofs, for instance, this is mentioned in the preface written by Felix Klein.

It is the purpose of this paper to supply the missing proofs for one of the asymmetric cyclidic coordinate systems which is listed as number 12 in Miller’s list (see p. 210 of Ref. 9) (see also...
FIG. 1. Surfaces \( s_1, s_2, s_3 = \text{const} \) for \( a_i = i \), where only the component of the cyclide \( s_1 = \text{const} \) inside the ball \( x^2 + y^2 + z^2 < 1 \) is shown.

Table II of Ref. 3 and Ref. 4 for a more general setting). For lack of a better name we call it 5-cyclide coordinates. This asymmetric orthogonal curvilinear coordinate system has coordinates \( s_i \in \mathbb{R} \) \( (i = 1, 2, 3) \) with \( s_i \) in \((a_0, a_1)\), \((a_1, a_2)\) or \((a_2, a_3)\), respectively, where \( a_0 < a_1 < a_2 < a_3 \) are given numbers. This coordinate system is described by coordinate surfaces \( s_i = \text{const} \) which are five compact cyclides. The surfaces \( s_1 = \text{const} \) for \( s_1 \in (a_0, a_1) \) are two cyclides of genus zero representing inversions of each other with respect to the unit sphere. The surface \( s_2 = \text{const} \) for \( s_1 \in (a_1, a_2) \) represents a ring cyclide of genus one and the surfaces \( s_3 = \text{const} \) for \( s_1 \in (a_2, a_3) \) represent two disconnected cyclides of genus zero with reflection symmetry about the \( x, y \)-plane. The asymptotic behavior of this coordinate system as the size of these compact cyclides increases without limit is 6-sphere coordinates (see p. 122 of Ref. 10), the inversion of Cartesian coordinates.

In our notation, the coordinate surfaces of this system are given by the variety

\[
\frac{(x^2 + y^2 + z^2 - 1)^2}{s - a_0} + \frac{4x^2}{s - a_1} + \frac{4y^2}{s - a_2} + \frac{4z^2}{s - a_3} = 0, \tag{2}
\]

where \( s = s_i \) is either in \((a_0, a_1)\), \((a_1, a_2)\), or \((a_2, a_3)\), respectively.

See Figures 1(a) and 1(b) for a graphical illustration of these triply orthogonal coordinate surfaces, where we have selected one of the confocal cyclides for \( s_1 = \text{const} \). This is a very general coordinate system containing the parameters \( a_0, a_1, a_2, a_3 \) which generates many other coordinate systems by limiting processes. For example, rotationally invariant flat-ring coordinates (System 15 on p. 210 of Ref. 9) are obtained by setting \( s_3 = a_2 \sin^2 \phi + a_3 \cos^2 \phi \) and letting \( a_3 \to a_2 \), and rotationally invariant bicyclidic coordinates (System 14 on p. 210 of Ref. 9) are obtained by setting \( s_2 = a_1 \sin^2 \phi + a_2 \cos^2 \phi \) and letting \( a_2 \to a_1 \). Since the book by Böcher is quite old and uses very geometrical methods, we will present our results independently of Böcher’s book. We supply convergence proofs based on general multiparameter spectral theory \(^{1,13} \) which was created with such applications in mind. As far as we know this general theory has never before been applied to the Dirichlet problems considered by Böcher.

We start with the observation that 5-cyclide coordinates are the stereographic image of spheroidal coordinates in four dimensions (or, expressed in another way, of ellipsoidal coordinates on the hypersphere \( S^3 \)). We take the spheroidal coordinate system as known but we present the needed facts in Sec. II. The well-known stereographic projection is dealt with in Sec. III which also explains the appearance of the factor \( R \) in (1). The 5-cyclide coordinate system is introduced in Sec. IV. The solution of the Dirichlet problem on regions bounded by surfaces (2) with \( s \in (a_1, a_2) \) is presented in Sec. VI. Section V provides the needed convergence proofs based on multiparameter spectral theory. The remaining Secs. VII–X treat the Dirichlet problem on regions bounded by the surfaces (2) when \( s \in (a_1, a_2) \) (ring cyclides) and \( s \in (a_2, a_3) \).
II. SPHERO-CONAL COORDINATES ON $\mathbb{R}^{k+1}$

Let $k \in \mathbb{N}$. In order to introduce sphero-conal coordinates on $\mathbb{R}^{k+1}$, fix real numbers

$$a_0 < a_1 < a_2 < \cdots < a_k.$$  

(3)

Let $(x_0, x_1, \ldots, x_k)$ be in the positive cone of $\mathbb{R}^{k+1}$

$$x_0 > 0, \ldots, x_k > 0.$$  

(4)

Its sphero-conal coordinates $r, s_1, \ldots, s_k$ are determined in the intervals

$$r > 0, \ a_{i-1} < s_i < a_i, \ i = 1, \ldots, k$$  

(5)

by the equations

$$r^2 = \sum_{j=0}^{k} x_j^2$$  

(6)

and

$$\sum_{j=0}^{k} \frac{x_j^2}{s_i - a_j} = 0 \quad \text{for } i = 1, \ldots, k.$$  

(7)

The latter equation determines $s_1, s_2, \ldots, s_k$ as the zeros of a polynomial of degree $k$ with coefficients which are polynomials in $x_0^2, \ldots, x_k^2$.

In this way, we obtain a bijective (real-)analytic map from the positive cone in $\mathbb{R}^{k+1}$ to the set of points $(r, s_1, \ldots, s_k)$ satisfying (5). The inverse map is found by solving a linear system. It is also analytic, and it is given by

$$x_j^2 = r^2 \frac{\prod_{i=1}^{k} (s_i - a_j)}{\prod_{j \neq i=0}^{k} (a_i - a_j)}.$$  

(8)

Sphero-conal coordinates are orthogonal, and its scale factors (metric coefficients) are given by

$$H_r = 1,$$

$$H_{s_i}^2 = \frac{1}{4} \sum_{j=0}^{k} \frac{x_j^2}{(s_i - a_j)^2} = -\frac{1}{4} \frac{\prod_{j \neq i=1}^{k} (s_j - s_j)}{\prod_{j=0}^{i} (s_i - a_j)}, \quad i = 1, 2, \ldots, k.$$  

(9)

Consider the Laplace equation

$$\Delta U = \sum_{i=0}^{k} \frac{\partial^2 U}{\partial x_i^2} = 0$$  

(10)

for a function $U(x_0, x_1, \ldots, x_k)$. Using (9) we transform this equation to sphero-conal coordinates, and then we apply the method of separation of variables

$$U(x_0, x_1, \ldots, x_k) = w_0(r) w_1(s_1) w_2(s_2) \cdots w_k(s_k).$$  

(11)

For the variable $r$ we obtain the Euler equation

$$w_0'' + \frac{k}{r} w_0' + \frac{4\lambda_0}{r^2} w_0 = 0,$$  

(12)

while for each of the variables $s_1, s_2, \ldots, s_k$, we obtain the Fuchsian equation

$$\prod_{j=0}^{k} (s - a_j) \left[ w'' + \frac{1}{2} \sum_{j=0}^{k} \frac{1}{s - a_j} w' \right] + \sum_{i=0}^{k-1} \lambda_i s^{k-1-i} w = 0.$$  

(13)

More precisely, if $\lambda_0, \ldots, \lambda_{k-1}$ are any given numbers (separation constants), and if $w_0(r), r > 0$, solves (12) and $w_j(s_i), a_{i-1} < s_i < a_i$, solve (13) for each $i = 1, \ldots, k$, then $U$ defined by (11) solves (10) in the positive cone of $\mathbb{R}^{k+1}$ (4).
Equation (13) has only regular points except for \( k + 2 \) regular singular points at \( s = a_0, a_1, \ldots, a_k \) and \( s = \infty \). The exponents at each finite singularity \( s = a_j \) are 0 and \( \frac{1}{2} \). Therefore, for each choice of parameters \( \lambda_0, \ldots, \lambda_k - 1 \), there is a nontrivial analytic solution at \( s = a_j \) and another one of the form \( w(s) = (s - a_j)^{1/2} v(s) \), where \( v \) is analytic at \( a_j \). If \( v, \mu \) denote the exponents at \( s = \infty \), then

\[
\mu + v = \frac{k - 1}{2}.
\] (14)

The polynomial \( \sum_{i=0}^{k-1} \lambda_i s^{k-1-i} \) appearing in (13) is known as van Vleck polynomial. If \( k = 1 \), then (13) is the hypergeometric differential equation (up to a linear substitution). If \( k = 2 \), then (13) is the Heun equation. We will use this equation for \( k = 3 \). According to Miller (see p. 209 of Ref. 9) (see also p. 71 of Ref. 3) in reference to the \( k = 3 \) case, “Very little is known about the solutions.”

III. STEREOGRAPHIC PROJECTION

We consider the stereographic projection \( P : \mathbb{S}^3 \setminus \{(1, 0, 0, 0)\} \rightarrow \mathbb{R}^3 \) given by

\[
P(x_0, x_1, x_2, x_3) = \frac{1}{1-x_0}(x_1, x_2, x_3).
\]

The inverse map is

\[
P^{-1}(x, y, z) = \frac{1}{x^2 + y^2 + z^2 + 1}(x^2 + y^2 + z^2 - 1, 2x, 2y, 2z).
\]

We extend \( P^{-1} \) to a bijective map

\[
Q : (0, \infty) \times \mathbb{R}^3 \rightarrow \mathbb{R}^4 \setminus \{(x_0, 0, 0, 0) : x_0 \geq 0\}
\]

by defining

\[
Q(r, x, y, z) := rP^{-1}(x, y, z).
\]

If we set \((x_0, x_1, x_2, x_3) = Q(r, x, y, z)\), we may consider \( r, x, y, z \) as curvilinear coordinates on \( \mathbb{R}^4 \) with Cartesian coordinates \( x_0, x_1, x_2, x_3 \). We note that \( x_0^2 + x_1^2 + x_2^2 + x_3^2 = r^2 \) so \( r \) is just the distance between \((x_0, x_1, x_2, x_3)\) and the origin. Moreover, \((x, y, z)\) is the stereographic projection of the point \((x_0/r, x_1/r, x_2/r, x_3/r) \in \mathbb{S}^3\). It is easy to check that the coordinate system is orthogonal and scale factors are

\[
h_r = 1, \quad h_x = h_y = h_z = 2rh, \quad \text{where} \quad h := \frac{1}{x^2 + y^2 + z^2 + 1}.
\]

Let \( U(x_0, x_1, x_2, x_3) = V(r, x, y, z) \). Then

\[
\Delta U = \frac{1}{8r^3h^3} \left( (2rhV_x)_x + (2rhV_y)_y + (2rhV_z)_z + (8r^3h^3V)_{rr} \right).
\] (15)

Suppose that \( U \) is homogeneous of degree \( \alpha \):

\[
U(tx_0, tx_1, tx_2, tx_3) = t^\alpha U(x_0, x_1, x_2, x_3), \quad t > 0.
\]

Then \( V \) can be written in the form

\[
V(r, x, y, z) = r^\alpha w(x, y, z),
\]

and (15) implies

\[
\Delta U = \frac{r^{\alpha-2}}{4h^3} \left( (hw_x)_x + (hw_y)_y + (hw_z)_z + 4\alpha(\alpha + 2)h^3 w \right).
\] (16)

We now introduce the function

\[
u(x, y, z) = w(x, y, z)(x^2 + y^2 + z^2 + 1)^{-1/2}.
\]
Then a direct calculation changes (16) to
\[
\Delta U = \frac{r^\alpha - 2}{4\sqrt{h}^3} (u_{xx} + u_{yy} + u_{zz} + (3 + 4\alpha(\alpha + 2))h^2u). \tag{17}
\]
If \(3 + 4\alpha(\alpha + 2) = 0\), then \(U\) is harmonic if and only if \(u\) is harmonic. Noting that \(3 + 4\alpha(\alpha + 2) = (2\alpha + 1)(2\alpha + 3)\), we obtain the following theorem.

**Theorem 3.1.** Let \(D\) be an open subset of \(S^3\) not containing \((1, 0, 0, 0)\), let \(E = \{(r_0, r_1, r_2, r_3) : r > 0, (x_0, x_1, x_2, x_3) \in D\}\), and let \(F = P(D)\) be the stereographic image of \(D\). Let the function \(U: E \to \mathbb{R}\) be homogeneous of degree \(-\frac{1}{2}\) or \(-\frac{3}{2}\), and let \(w : F \to \mathbb{R}\) satisfy \(U = w \circ P\) on \(D\). Then \(U\) is harmonic on \(E\) if and only if \(w(x, y, z)(x^2 + y^2 + z^2 + 1)^{-1/2}\) is harmonic on \(F\).

**IV. FIVE-CYCLIDE COORDINATE SYSTEM ON \(R^3\)**

We introduce sphero-conal coordinates
\[
r > 0, \quad a_0 < s_1 < a_1 < s_2 < a_2 < s_3 < a_3,
\]
on \(R^4\) as explained in Sec. II with \(k = 3\). Then \(s_1, s_2, s_3\) form a coordinate system for the intersection of the hypersphere \(S^3\) with the positive cone in \(R^4\). Using the stereographic projection \(P\) from Sec. III, we project these coordinates to \(R^3\). We obtain a coordinate system for the set
\[
T = \{(x, y, z) : x, y, z > 0, x^2 + y^2 + z^2 > 1\}. \tag{18}
\]
Explicitly,
\[
x = \frac{x_1}{1 - x_0}, \quad y = \frac{x_2}{1 - x_0}, \quad z = \frac{x_3}{1 - x_0}, \tag{19}
\]
where
\[
x_j^2 = \frac{\prod_{i=1}^3(s_i - a_j)}{\prod_{i,j \neq 0}(a_i - a_j)}, \quad j = 0, 1, 2, 3. \tag{20}
\]
Conversely, the coordinates \(s_1, s_2, s_3\) of a point \((x, y, z) \in T\) are the solutions of
\[
\frac{(x^2 + y^2 + z^2 - 1)^2}{s - a_0} + \frac{4x^2}{s - a_1} + \frac{4y^2}{s - a_2} + \frac{4z^2}{s - a_3} = 0. \tag{21}
\]
Since sphero-conal coordinates are orthogonal and the stereographic projection preserves angles, 5-cyclide coordinates are orthogonal, too. This is the twelfth coordinate system in Miller (see p. 210 of Ref. 9). Miller uses a slightly different notation: \(a_0 = 0, a_1 = 1, a_2 = b, a_3 = a, s_1 = \rho, s_2 = v, s_3 = \mu\). Also, \(x, z\) are interchanged.

In order to calculate the scale factors for the 5-cyclide coordinate system, we proceed as follows. We start with
\[
\frac{\partial x}{\partial s_i} = \frac{1}{1 - x_0} \frac{\partial x_1}{\partial s_i} + \frac{x_1}{(1 - x_0)^2} \frac{\partial x_0}{\partial s_i},
\]
and similar formulas for the derivatives of \(y\) and \(z\). Then using
\[
x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1, \quad x_0 \frac{\partial x_0}{\partial s_i} + x_1 \frac{\partial x_1}{\partial s_i} + x_2 \frac{\partial x_2}{\partial s_i} + x_3 \frac{\partial x_3}{\partial s_i} = 0,
\]
a short calculation gives
\[
\frac{\partial x}{\partial s_i} \frac{\partial x}{\partial s_j} + \frac{\partial y}{\partial s_i} \frac{\partial y}{\partial s_j} + \frac{\partial z}{\partial s_i} \frac{\partial z}{\partial s_j} = \frac{1}{(1 - x_0)^2} \sum_{i=0}^3 \frac{\partial x_i \partial x_i}{\partial s_i \partial s_j}.
\]
This confirms that 5-cyclide coordinates are orthogonal and from (9) we obtain the squares of their scale factors
\[
h_i^2 = \frac{1}{16} \left( \frac{(\rho^2 - 1)^2}{(s_i - a_0)^2} + \frac{4\rho^2}{(s_i - a_1)^2} + \frac{4\rho^2}{(s_i - a_2)^2} + \frac{4\rho^2}{(s_i - a_3)^2} \right), \tag{22}
\]
where \( \rho^2 = x^2 + y^2 + z^2 \), or, equivalently,

\[
\begin{align*}
  h_1^2 &= \frac{1}{16} (\rho^2 + 1)^2 \frac{(s_3 - s_1)(s_2 - s_1)}{(s_1 - a_0)(a_1 - s_1)(a_2 - s_1)(a_3 - s_1)}, \\
  h_2^2 &= \frac{1}{16} (\rho^2 + 1)^2 \frac{(s_2 - s_1)(s_3 - s_2)}{(s_2 - a_0)(s_2 - a_1)(a_2 - s_2)(a_3 - s_2)}, \\
  h_3^2 &= \frac{1}{16} (\rho^2 + 1)^2 \frac{(s_3 - s_1)(s_3 - s_2)}{(s_3 - a_0)(s_3 - a_1)(s_3 - a_2)(a_3 - s_3)}.
\end{align*}
\]

(23) \hspace{1cm} (24) \hspace{1cm} (25)

We find harmonic functions by separation of variables in 5-cyclide coordinates as follows.

**Theorem 4.1.** Let \( w_1 : (a_0, a_1) \to \mathbb{C}, w_2 : (a_1, a_2) \to \mathbb{C}, w_3 : (a_2, a_3) \to \mathbb{C} \) be solutions of the Fuchsian equation

\[
\prod_{j=0}^{3} (s - a_j) \left[ w'' + \frac{1}{2} \sum_{j=0}^{3} \frac{s - a_j}{s - a_j} w' \right] + \left( \frac{3}{16} s^2 + \lambda_1 s + \lambda_2 \right) w = 0,
\]

(26)

where \( \lambda_1, \lambda_2 \) are given (separation) constants. Then the function

\[
w(x, y, z) = (x^2 + y^2 + z^2 + 1)^{-1/2} w_1(s_1) w_2(s_2) w_3(s_3)
\]

(27)

is a harmonic function on the set (18).

**Proof.** Using sphero-conal coordinates \( r, s, t, s_3 \) on \( \mathbb{R}^4 \), we define a function \( U \) in the positive cone of \( \mathbb{R}^4 \) by

\[
U(x_0, x_1, x_2, x_3) = r^{-1/2} w_1(s_1) w_2(s_2) w_3(s_3).
\]

The function \( r^{-1/2} \) is a solution of (12) when \( k = 3, \lambda_0 = \frac{3}{16} \). The results from Sec. II imply that \( U \) is harmonic, and, of course, \( U \) is homogeneous of degree \(-\frac{1}{2}\). The function \( w \) defined on the set (18) by \( U = w \circ P \) is given in 5-cycloid coordinates by

\[
w(x, y, z) = w_1(s_1) w_2(s_2) w_3(s_3).
\]

Therefore, Theorem 3.1 gives the statement of the theorem. \( \square \)

Equation (26) has five regular singularities at \( s = a_0, a_1, a_2, a_3, \infty \). The exponents at the finite singularities are 0 and \( \frac{1}{2} \). Using (14), we find that the exponents at infinity are \( \frac{3}{4} \) and \( \frac{3}{4} \). So all five singularities are elementary in the sense of Ince.\(^5\) Equation (26) is one of the standard equations in the classification of Ince (see p. 500 of Ref. 5).

We define the 5-cyclide coordinates \( s_1, s_2, s_3 \) for an arbitrary point \( (x, y, z) \in \mathbb{R}^3 \) as the zeros \( s_1 \leq s_2 \leq s_3 \) of the cubic equation

\[
\prod_{j=0}^{3} (s - a_j) \left[ \frac{(x^2 + y^2 + z^2 - 1)^2}{s - a_0} + \frac{4x^2}{s - a_1} + \frac{4y^2}{s - a_2} + \frac{4z^2}{s - a_3} \right] = 0.
\]

(28)

For example, \( s_j(0, 0, 0) = a_j \) for \( j = 1, 2, 3 \). Each function \( s_j: \mathbb{R}^3 \to [a_j - 1, a_i] \) is continuous. We observe that, in general, there are 16 different points in \( \mathbb{R}^3 \) which have the same coordinates \( s_1, s_2, s_3 \). If \( (x, y, z) \) is one of these points, the other ones are obtained by applying the group generated by inversion at \( S^2 \)

\[
s_0(x, y, z) = \rho^{-2}(x, y, z)
\]

(29)

and reflections at the coordinate planes

\[
s_1(x, y, z) = (-x, y, z), \ s_2(x, y, z) = (x, -y, z), \ s_3(x, y, z) = (x, y, -z).
\]

(30)

It is of interest to determine the sets where \( s_j = a_j - 1 \) or \( s_j = a_j \). We obtain

\[
s_1 = a_0 \text{ iff } x^2 + y^2 + z^2 = 1.
\]

(31)
The set $A_1$ in the $y, z$-plane for $a_i = i$. 

We define the sets (consisting each of two closed curves)

\[ A_1 := \{(x, y, z) \in \mathbb{R}^3 : s_1 = s_2 = a_1\} \]

\[ = \{(x, y, z) : x = 0, (\rho^2 - 1)^2 \frac{a^2}{a_1 - a_0} + 4\frac{y^2}{a_1 - a_2} + 4\frac{z^2}{a_1 - a_3} = 0\}, \] 

see Figure 2, and

\[ A_2 := \{(x, y, z) \in \mathbb{R}^3 : s_2 = s_3 = a_2\} \]

\[ = \{(x, y, z) : y = 0, (\rho^2 - 1)^2 \frac{a^2}{a_2 - a_0} + 4\frac{x^2}{a_2 - a_1} + 4\frac{z^2}{a_2 - a_3} = 0\}, \]

see Figure 3. Clearly, $s_j$ is analytic at all points $(x, y, z)$ at which $s_j$ is a simple zero of the cubic equation (28). Therefore, $s_1$ is analytic on $\mathbb{R}^3 \setminus A_1$, $s_2$ is analytic on $\mathbb{R}^3 \setminus (A_1 \cup A_2)$, and $s_3$ is analytic on $\mathbb{R}^3 \setminus A_2$.

We may use (27) to define $u(x, y, z)$ for all $(x, y, z) \in \mathbb{R}^3$. Since the solutions $w_1, w_2, w_3$ of (26) have limits at the end points of their intervals of definition (because the exponents are 0 and $\frac{1}{2}$ there), we see that $u$ is a continuous function on $\mathbb{R}^3$. The function $(x^2 + y^2 + z^2 + 1)^{1/2}u(x, y, z)$
FIG. 3. The set $A_2$ in the $x,z$-plane for $a_i = i$.

is invariant under $\sigma_i$, $i = 0, 1, 2, 3$. In general, $u$ is harmonic only away from the coordinate planes and the unit sphere. In fact, we observe that $u$ is a bounded function which converges to 0 at infinity, so, by Liouville’s theorem, $u$ cannot be harmonic on all of $\mathbb{R}^3$ unless it is identically zero.

V. FIRST TWO-PARAMETER STURM-LIOUVILLE PROBLEM

We consider equation (26) on the intervals $(a_1, a_2)$ and $(a_2, a_3)$ and write it in formally self-adjoint form. Setting

$$\omega(s) := |(s - a_0)(s - a_1)(s - a_2)(s - a_3)|^{1/2},$$

(39)
we obtain two Sturm-Liouville equations involving two parameters
\[
(\omega(s_2)w_2')' + \frac{1}{\omega(s_2)} \left( \frac{3}{16} s_2^2 + \lambda_1 s_2 + \lambda_2 \right) w_2 = 0, \quad a_1 < s_2 < a_2, \tag{40}
\]
\[
(\omega(s_3)w_3')' - \frac{1}{\omega(s_3)} \left( \frac{3}{16} s_3^2 + \lambda_1 s_3 + \lambda_2 \right) w_3 = 0, \quad a_2 < s_3 < a_3. \tag{41}
\]
In (40), \( w_2 \) is a function of \( s_2 \) and the derivatives are taken with respect to \( s_2 \). In (41), \( w_3 \) is a function of \( s_3 \) and the derivatives are taken with respect to \( s_3 \). We simplify the equations by substituting \( t_j = \Omega(s_j) \), \( u_j(t_j) = w_j(s_j) \), where \( \Omega(s) \) is the elliptic integral (see, for instance, Ref. 8)
\[
\Omega(s) := \int_{a_0}^s \frac{d\sigma}{\omega(\sigma)}. \tag{42}
\]
This is an increasing absolutely continuous function \( \Omega: [a_0, a_3] \to [0, b_3] \), where \( b_j := \Omega(a_j) \). Let \( \phi: [0, b_3] \to [a_0, a_3] \) be the inverse function of \( \Omega \). Then (40) and (41) become
\[
u''_2 + \left( \frac{3}{16} (\phi(t_2))^2 + \lambda_1 \phi(t_2) + \lambda_2 \right) u_2 = 0, \quad b_1 < t_2 < b_2, \tag{43}
\]
\[
u''_3 - \left( \frac{3}{16} (\phi(t_3))^2 + \lambda_1 \phi(t_3) + \lambda_2 \right) u_3 = 0, \quad b_2 < t_3 < b_3. \tag{44}
\]
We add the boundary conditions
\[
u'(b_1) = u''_2(b_2) = u'_3(b_3) = 0. \tag{45}
\]

Differential equations (43) and (44) together with boundary conditions (45) pose a two-parameter Sturm-Liouville eigenvalue problem. For the theory of such multiparameter problems, we refer to the studies\cite{1,13} and the references therein. A pair \((\lambda_1, \lambda_2)\) is called an eigenvalue if there exist (nontrivial) eigenfunctions \( u_2(t_2) \) and \( u_3(t_3) \) which satisfy (43)–(45). The two-parameter problem is right-definite in the sense that
\[
\begin{vmatrix}
\phi(t_2) & 1 \\
-\phi(t_3) & 1
\end{vmatrix} = \phi(t_3) - \phi(t_2) > 0 \quad \text{for } b_1 < t_2 < b_2 < t_3 < b_3.
\]
However, this determinant is not positive on the closed rectangle \([b_1, b_2] \times [b_2, b_3]\). This lack of uniform right-definiteness make some proofs in this section a little longer than they would be otherwise.

We have the following Klein oscillation theorem (see Theorem 5.5.1 of Ref. 1).

**Theorem 5.1.** For every \( n = (n_2, n_3) \in \mathbb{N}_0^2 \), there exists a uniquely determined eigenvalue \((\lambda_{1,n}, \lambda_{2,n}) \in \mathbb{R}^2\) admitting an eigenfunction \( u_2 \) with exactly \( n_2 \) zeros in \((b_1, b_2)\) and an eigenfunction \( u_3 \) with exactly \( n_3 \) zeros in \((b_2, b_3)\).

We state a result on the distribution of eigenvalues (compare with Chap. 8 of Ref. 1).

**Theorem 5.2.** There are positive constants \( A_1, A_2, A_3, A_4 \) such that, for all \( n \in \mathbb{N}_0^2 \),
\[
-A_1(n_2^2 + n_3^2 + 1) \leq \lambda_{1,n} \leq -A_2(n_2^2 + n_3^2) + A_3, \tag{46}
\]
\[
|\lambda_{2,n}| \leq A_4(n_2^2 + n_3^2 + 1). \tag{47}
\]

**Proof.** If a differential equation \( u'' + q(t)u = 0 \) with continuous \( q: [a, b] \to \mathbb{R} \) admits a solution \( u \) satisfying \( u''(a) = u'(b) = 0 \) and having exactly \( m \) zeros in \((a, b)\), then there is \( t \in (a, b) \) such that \( q(t) = \frac{\pi^2 m^2}{(b-a)^2} \). This is shown by comparing with the eigenvalue problem \( u'' + \lambda u = 0 \),
By Sturm’s comparison theorem applied to Eq. (43), we get
\[
\frac{3}{16} (\phi(t_3))^2 + \lambda_1 \phi(t_3) + \lambda_2 = \frac{\pi^2 n^2_3}{(b_3 - b_2)^2}.
\]
where we abbreviated \( \lambda_j = \lambda_{j,n} \). By subtracting (48) from (49), we obtain
\[
\frac{3}{16} (\phi(t_3))^2 - (\phi(t_2))^2 + \lambda_1 (\phi(t_3) - \phi(t_2)) = -\frac{\pi^2 n^2_2}{(b_2 - b_1)^2} - \frac{\pi^2 n^2_3}{(b_3 - b_2)^2} \leq 0.
\]
Dividing by \( \phi(t_3) - \phi(t_2) \) and using \( 0 < \phi(t_3) - \phi(t_2) \leq a_3 - a_1 \), we obtain the second inequality in (46).

To prove the first inequality in (46), suppose that \( \lambda_1 < -\frac{3}{8} a_3 \). Then the van Vleck polynomial
\[
Q(s) := \frac{3}{16} s^2 + \lambda_1 s + \lambda_2
\]
satisfies \( Q'(s) = \frac{3}{8} s + \lambda_1 < 0 \) for \( s \leq a_3 \). Let \( c \in (b_1, b_2) \) be determined by \( \phi(c) = \frac{1}{2} (a_1 + a_2) \). If \( Q(a_2) \geq 0 \) then, for \( t \in [b_1, c] \),
\[
Q(\phi(t)) \geq Q\left(\frac{1}{2} (a_1 + a_2)\right) \geq \frac{1}{2} (a_2 - a_1) \left(-\lambda_1 - \frac{3}{8} a_3\right).
\]
By Sturm’s comparison theorem applied to Eq. (43), we get
\[
(c - b_1)^2 (a_2 - a_1) \left(-\lambda_1 - \frac{3}{8} a_3\right) \leq 4 \pi^2 (a_2 + 1)^2,
\]
which gives the desired inequality. If \( Q(a_2) < 0 \), we argue similarly working with (44) instead.

Finally, (47) follows from (46) and (48). \( \boxdot \)

Let \( u_{2,n} \) and \( u_{3,n} \) denote real-valued eigenfunctions corresponding to the eigenvalue \( (\lambda_{1,n}, \lambda_{2,n}) \).

It is known (see Sec. 3.5 of Ref. 1) (and easy to prove) that the system of products \( u_{2,n}(t_2) u_{3,n}(t_3) \), \( n \in \mathbb{N}_0 \), is orthogonal in the Hilbert space \( H_1 \) consisting of measurable functions \( f : (b_1, b_2) \times (b_2, b_3) \to \mathbb{C} \) satisfying
\[
\int_{b_1}^{b_2} \int_{b_1}^{b_2} (\phi(t_3) - \phi(t_2)) |f(t_2, t_3)|^2 \, dt_2 \, dt_3 < \infty
\]
with inner product
\[
\int_{b_1}^{b_2} \int_{b_1}^{b_2} (\phi(t_3) - \phi(t_2)) f(t_2, t_3) \overline{g(t_2, t_3)} \, dt_2 \, dt_3.
\]
We normalize the eigenfunctions so that
\[
\int_{b_1}^{b_2} \int_{b_1}^{b_2} (\phi(t_3) - \phi(t_2)) \left[ u_{2,n}(t_2) \right]^2 \left[ u_{3,n}(t_3) \right]^2 \, dt_2 \, dt_3 = 1.
\]
(51)

We have the following completeness theorem (see Theorem 6.8.3 of Ref. 13).

**Theorem 5.3.** The double sequence of functions \( u_{2,n}(t_2) u_{3,n}(t_3), \ n \in \mathbb{N}_0^2 \),
forms an orthonormal basis in the Hilbert space \( H_1 \).

The normalization (51) leads to a bound on the values of eigenfunctions.
Theorem 5.4. There is a constant $B > 0$ such that, for all $n \in \mathbb{N}_0^2$ and all $t_2 \in [b_1, b_2]$, $t_3 \in [b_2, b_3]$,

$$|u_{2,n}(t_2)u_{3,n}(t_3)| \leq B(n_2^2 + n_3^2 + 1).$$

Proof. We abbreviate $u_j = u_{j,n}$, $\lambda_j = \lambda_{j,n}$. Condition (51) is a normalization for the product $u_2(t_2)u_3(t_3)$ but not for each factor separately, so we may assume that, additionally,

$$\int_{b_1}^{b_2} (u_2(t_2))^2 dt_2 = 1. \quad (52)$$

Now (51) and (52) imply that

$$\int_{b_2}^{b_3} (\phi(t_3) - \phi(b_2)) |u_3(t_3)|^2 dt_3 \leq 1. \quad (53)$$

We multiply Eqs. (43) and (44) by $u_2$ and $u_3$, respectively, and integrate by parts to obtain

$$\int_{b_1}^{b_2} u_2^2 = \frac{3}{16} \int_{b_1}^{b_2} \phi^2 u_2^2 + \lambda_1 \int_{b_1}^{b_2} \phi u_2^2 + \lambda_2 \int_{b_1}^{b_2} u_2^2, \quad (54)$$

$$\int_{b_2}^{b_3} u_3^2 = -\frac{3}{16} \int_{b_2}^{b_3} \phi^2 u_3^2 - \lambda_1 \int_{b_2}^{b_3} \phi u_3^2 - \lambda_2 \int_{b_2}^{b_3} u_3^2. \quad (55)$$

It follows from (52) and (54) and Theorem 5.2 that there is a constant $B_1 > 0$ such that, for all $n \in \mathbb{N}_0^2$,

$$\int_{b_1}^{b_2} u_2^2 \leq B_1(n_2^2 + n_3^2 + 1). \quad (56)$$

Unfortunately, we cannot argue the same way for $u_3$ because we do not have an upper bound for $\int_{b_2}^{b_3} u_2^2$. Instead, we multiply (54) by $\int u_3^2$ and (55) by $\int u_2^2$ and add the equations. Then, noting (51), we find

$$\int_{b_1}^{b_2} u_2^2 \int_{b_2}^{b_3} u_3^2 + \int_{b_1}^{b_2} u_2^2 \int_{b_2}^{b_3} u_3^2 \leq -\lambda_1 + \frac{3}{8} \max_{t \in (b_1, b_3)} |\phi(t)|. \quad (57)$$

Using Theorem 5.2 and (52), we find a constant $B_2 > 0$ such that, for all $n \in \mathbb{N}_0^2$,

$$\int_{b_2}^{b_3} u_3^2 \leq B_2(n_2^2 + n_3^2 + 1). \quad (57)$$

We apply the following Lemma 5.5 (noting (52), (53), (56), (57)) and obtain the desired result.

Lemma 5.5. Let $u: [a, b] \to \mathbb{R}$ be a continuously differentiable function, and let $a \leq c < d \leq b$. Then, for all $t \in [a, b]$,

$$(d - c)|u(t)|^2 \leq 2 \int_c^d |u(r)|^2 dr + 2(b - a)(d - c) \int_a^b |u'(r)|^2 dr.$$

Proof. For $s, t \in [a, b]$, we have

$$|u(t) - u(s)| = \left| \int_s^t u'(r) dr \right| \leq |t - s|^{1/2} \left( \int_a^b |u'(r)|^2 dr \right)^{1/2}.$$

This implies

$$|u(t)|^2 \leq 2|u(s)|^2 + 2|t - s| \int_a^b |u'(r)|^2 dr.$$

We integrate from $s = c$ to $s = d$ and obtain the desired inequality.
Let \( u_{1,n} \) be the solution of
\[
 u'' - \left( \frac{3}{16} (\phi(t_1))^2 + \lambda_{1,n} \phi(t_1) + \lambda_{2,n} \right) u_1 = 0, \quad b_0 \leq t_1 \leq b_1, \tag{58}
\]
determined by the initial conditions
\[
 u_1(b_1) = 1, \quad u'_1(b_1) = 0. \tag{59}
\]
The following estimate on \( u_{1,n} \) will be useful in Sec. VI.

**Theorem 5.6.** We have \( u_{1,n}(t_1) > 0 \) for all \( t_1 \in [b_0, b_1] \). If \( 0 = b_0 \leq c_1 < c_2 < b_1 \), then there are constants \( C > 0 \) and \( 0 < r < 1 \) such that, for all \( n \in \mathbb{N}_0^2 \) and \( t_1 \in [c_2, b_1] \),
\[
 \frac{u_{1,n}(t_1)}{u_{1,n}(c_1)} \leq C r^{n_2+n_3}. \tag{60}
\]

**Proof.** We abbreviate \( u = u_{1,n} \) and \( \lambda_j = \lambda_{j,n} \). By definition, \( u_1 \) satisfies the differential equation
\[
 u'' = Q(\phi(t_1))u_1, \quad t_1 \in [b_0, b_1],
\]
where \( Q \) is given by (50). According to (48) and (49), there are \( s_2 \in (a_1, a_2) \) and \( s_3 \in (a_2, a_3) \) such that
\[
 Q(s_2) = \frac{\pi^2 n_2^2}{(b_2 - b_1)^2}, \quad Q(s_3) = -\frac{\pi^2 n_3^2}{(b_3 - b_2)^2}. \tag{61}
\]
If \( s \leq s_2 \) then \( Q(s) \geq L(s) \), where \( L(s) \) is the linear function with \( L(s_j) = Q(s_j), j = 2, 3 \). It follows that \( Q(s) \geq 0 \) for \( s \in [a_0, a_1] \) and
\[
 Q(\phi(t_1)) \geq C_1(n_2 + n_3)^2 \quad \text{for} \quad t_1 \in [b_0, c_2], \tag{62}
\]
where \( C_1 \) is a positive constant independent of \( n \). We now apply the following Lemma 5.7 (with \( a = c_1, b = b_1, c = c_2 \)) to complete the proof. \( \square \)

**Lemma 5.7.** Let \( u: [a, b] \to \mathbb{R} \) be a solution of the differential equation
\[
 u''(t) = q(t)u(t), \quad t \in [a, b],
\]
determined by the initial conditions \( u(b) = 1, u'(b) = 0 \), where \( q: [a, b] \to \mathbb{R} \) is a continuous function. Suppose that \( q(t) \geq 0 \) on \( [a, b] \) and \( q(t) \geq \lambda^2 \) on \( [a, c] \) for some \( \lambda > 0 \) and \( c \in (a, b] \). Then \( u(t) > 0 \) for all \( t \in [a, b] \), and
\[
 \frac{u(t)}{u(a)} \leq 2e^{-\lambda(c-a)} \quad \text{for} \quad t \in [c, b]. \tag{63}
\]

**Proof.** Since \( q(t) \geq 0, u(t) > 0 \) and \( u'(t) \leq 0 \) for \( t \in [a, b] \). The function \( z = u'/u \) satisfies the Riccati equation
\[
 z' + z^2 = q(t),
\]
and the initial condition \( z(b) = 0 \). It follows that
\[
 z(t) \leq \lambda \tanh(\lambda(t-c)) \quad \text{for} \quad t \in [a, c].
\]
Integrating from \( t = a \) to \( t = c \) gives
\[
 \ln \frac{u(c)}{u(a)} \leq -\ln \cosh \lambda(c-a) \leq \ln(2e^{-\lambda(c-a)})
\]
which yields the claim since \( u \) is nonincreasing. \( \square \)

We now introduce a systematic notation for our eigenvalues and eigenfunctions. First of all, we note that the results of this section remain valid for other sets of boundary conditions. We will need
eight sets of boundary conditions labeled by $p = (p_1, p_2, p_3) \in \{0, 1\}^3$. These boundary conditions are

\begin{align*}
    u_0^p(b_1) &= 0 \quad \text{if } p_1 = 0, \quad u_2(b_1) = 0 \quad \text{if } p_1 = 1, \\
    u_1^p(b_2) &= u'_2(b_2) = 0 \quad \text{if } p_2 = 0, \quad u_2(b_2) = u_3(b_2) = 0 \quad \text{if } p_2 = 1, \\
    u_1^p(b_3) &= 0 \quad \text{if } p_3 = 0, \quad u_3(b_3) = 0 \quad \text{if } p_3 = 1.
\end{align*}

The initial conditions for $u_1$ are

\begin{align*}
    u_1(b_1) = 1, u'_1(b_1) = 0 \quad \text{if } p_1 = 0, \quad u_1(b_1) = 0, u'_1(b_1) = 1 \quad \text{if } p_1 = 1.
\end{align*}

We denote the corresponding eigenvalues by $(\lambda_{1,0,0,1}^{(1)}, \lambda_{2,0,0,1}^{(1)})$. For the notation of eigenfunctions, we return to the $s_i$-variable connected to $t_i$ by $t_i = \Omega(s_i)$. The eigenfunctions will be denoted by $E_{i,0,0,1}^{(1)}(s_i) = u_{i,0,0,1}(t_i)$, $i = 1, 2, 3$. The superscript (1) is used to distinguish from eigenvalues and eigenfunctions introduced in Secs. VII and IX. The subscript $n = (n_2, n_3)$ indicates the number of zeros of $E_{2,0,0,1}^{(1)}(s_2), E_{3,0,0,1}^{(1)}(s_3)$ in $(a_1, a_2), (a_2, a_3)$, respectively. The subscript $p$ indicates the boundary conditions used to determine eigenvalues and eigenfunctions. By using the letter $E$ for eigenfunctions, we follow Böcher.\(^2\) In our notation, we suppressed the dependence of eigenvalues and eigenfunctions on $a_0, a_1, a_2, a_3$.

Summarizing, for $i = 1, 2, 3$, $E_{i,0,0,1}^{(1)}$ is a solution of (26) on $(a_i-1, a_i)$ with $(\lambda_1, \lambda_2) = (\lambda_{1,0,0,1}^{(1)}, \lambda_{2,0,0,1}^{(1)})$. The solution $E_{1,0,0,1}^{(1)}(s_1)$ has exponent $\frac{1}{2}p_1$ at $a_1$ and it has no zeros in $(a_0, a_1)$.

The solution $E_{2,0,0,1}^{(1)}(s_2)$ has exponent $\frac{1}{2}p_1$ at $a_1$, exponent $\frac{1}{2}p_2$ at $a_2$, and it has $n_2$ zeros in $(a_1, a_2)$.

The solution $E_{3,0,0,1}^{(1)}(s_3)$ has exponent $\frac{1}{2}p_2$ at $a_2$, exponent $\frac{1}{2}p_3$ at $a_3$, and it has $n_3$ zeros in $(a_2, a_3)$.

VI. FIRST DIRICHLET PROBLEM

Consider the coordinate surface (21) for fixed $s = d_1 \in (a_0, a_1)$. See Figure 4 for a graphical depiction of the shape of this surface. Let $(x', y', z') \in S^2$. The ray $(x, y, z) = t(x', y', z'), t > 0$, intersects the surface if

\begin{align*}
    \frac{(t^2 - 1)^2}{d_1 - a_0} = ct^2,
\end{align*}

FIG. 4. Coordinate surfaces $s_1 = \text{const}$ for $a_i = i$ with (a), (b) inside $B_1(0)$; (d)–(f) outside $B_1(0)$; and (c) the unit sphere.
where
\[
c = \frac{4x^2}{a_1 - d_1} + \frac{4y^2}{a_2 - d_1} + \frac{4z^2}{a_3 - d_1} > 0.
\]
Equation (62) has two positive solutions \( t = t_1, t_2 \) such that \( 0 < t_1 < 1 < t_2 \) and \( t_1t_2 = 1 \). Therefore, the coordinate surface \( s_1 = d_1 \) consists of two disjoint closed surfaces of genus zero. One lies inside the unit ball \( B_1(0) \) centered at the origin and the other one is the image of it under the inversion (29).

Let \( D_1 \) be the region interior to the first surface, that is,
\[
D_1 = \{(x, y, z) \in B_1(0) : s_1 > d_1\},
\]
or, equivalently,
\[
D_1 = \{(x, y, z) \in B_1(0) : \frac{(\rho^2 - 1)^2}{d_1 - a_0} + \frac{4x^2}{d_1 - a_1} + \frac{4y^2}{d_1 - a_2} + \frac{4z^2}{d_1 - a_3} > 0\}.
\]
We showed that \( D_1 \) is star-shaped with respect to the origin. We now solve the Dirichlet problem for harmonic functions in \( D_1 \) by the method of separation of variables.

Let \( p = (p_1, p_2, p_3) \in [0, 1)^3 \) and \( n = (n_2, n_3) \in \mathbb{N}_0^2 \). Using the functions \( E_{i, n, p}^{(1)} \) introduced in Sec. V, we define the internal 5-cyclidic harmonic of the first kind
\[
G_{n, p}^{(1)}(x, y, z) = (x^2 + y^2 + z^2 + 1)^{-1/2} E_{1, n, p}^{(1)}(s_1) E_{2, n, p}^{(1)}(s_2) E_{3, n, p}^{(1)}(s_3)
\]
for \( x, y, z \in B_1(0) \) with \( x, y, z \geq 0 \). We extend this function to \( B_1(0) \) as a function of parity \( p \). We call a function \( f \) of parity \( p \) if
\[
f(\sigma_i(x, y, z)) = (-1)^{p_i} f(x, y, z), \quad \text{for } i = 1, 2, 3
\]
using the reflections (30).

Lemma 6.1. The function \( G_{n, p}^{(1)} \) is harmonic on \( B_1(0) \).

Proof. By Theorem 4.1, \( G_{n, p}^{(1)} \) is harmonic on \( B_1(0) \) away from the coordinate planes. Therefore, it is enough to show that \( G_{n, p}^{(1)} \) is analytic on \( B_1(0) \).

Consider first \( p = (0, 0, 0) \). Then (64) holds on \( B_1(0) \). Since \( s_1 \neq a_0 \) on \( B_1(0) \), \( G_{n, p}^{(1)} \) is analytic on \( B_1(0) \setminus (A_1 \cup A_2) \) as a composition of analytic functions. In order to show that \( G_{n, p}^{(1)} \) is also analytic at the points of \( A_1 \cup A_2 \), one may refer to a classical result on “singular curves” of harmonic functions (see Theorem XIII, p. 271 of Ref. 6), but we will argue more directly. Since \( A_1 \) and \( A_2 \) are disjoint sets, it is clear that \( E_{1, n, p}^{(1)}(s_1) \) is analytic at every point in \( B_1(0) \cap A_1 \). In order to show that \( E_{1, n, p}^{(1)}(s_1) E_{2, n, p}^{(1)}(s_2) \) is analytic at \( (x', y', z') \in B_1(0) \cap A_1 \), we argue as follows. We may assume that there is an analytic function \( w : (a_1, a_3) \to \mathbb{R} \) such that \( E_{1, n, p}^{(1)} \) and \( E_{2, n, p}^{(1)} \) are restrictions of this function to \((a_1, a_2)\) and \((a_2, a_3)\), respectively. Now \((s_1 - a_1) + (s_2 - a_1)\) and \((s_1 - a_1)(s_2 - a_1)\) are analytic functions of \((x, y, z)\) in a neighborhood of \((x', y', z')\). Lemma 6.2 implies that \( E_{1, n, p}^{(1)}(s_1) E_{2, n, p}^{(1)}(s_2) \) as a function of \((x, y, z)\) is analytic at \((x', y', z')\). It follows that \( G_{n, p}^{(1)} \) is analytic at every point in \( B_1(0) \cap A_1 \). In the same way, we show that \( G_{n, p}^{(1)} \) is analytic at every point in \( B_1(0) \cap A_2 \).

If \( p = (0, 0, 1) \), then we introduce the function
\[
\chi := \begin{cases} \sqrt{a_3 - s_3} & \text{if } z \geq 0 \\ -\sqrt{a_3 - s_3} & \text{otherwise}. \end{cases}
\]
It follows from (19) and (20) that \( \chi \) is analytic on \( \mathbb{R}^3 \setminus A_2 \). Then
\[
G_{n, p}^{(1)}(x, y, z) = (x^2 + y^2 + z^2 + 1)^{-1/2} E_{1, n, p}^{(1)}(s_1) E_{2, n, p}^{(1)}(s_2) \chi(x, y, z) w_3(s_3)
\]
on \( B_1(0) \), where \( w_3 \) is analytic at \( s_3 = a_3 \). We then argue as above.

The other parity vectors \( p \) are treated similarly. \[\Box\]
Lemma 6.2. Let \( f : (B_2)^2 \to \mathbb{C} \), \( B_2 = \{ s \in \mathbb{C} : |s| < \epsilon \} \), be an analytic function which is symmetric: \( f(s, t) = f(t, s) \). Let \( g, h : (B_3)^3 \to B_2 \) be functions such that \( g + h \) and \( gh \) are analytic. Then the function \( f(g(x, y, z), h(x, y, z)) \) is analytic on \( (B_3)^3 \).

Substituting \( t_j = \Omega(s_j) \), \( j = 2, 3 \), the Hilbert space \( H_1 \) from Sec. V transforms to the Hilbert space \( \overline{H}_1 \) consisting of measurable functions \( g : (a_1, a_2) \times (a_2, a_3) \to \mathbb{C} \) for which

\[
\|g\|^2 := \int_{a_1}^{a_2} \int_{a_1}^{a_2} \int_{a_1}^{a_2} \frac{s_3 - s_2}{\omega(s_2)\omega(s_3)} |g(s_2, s_3)|^2 ds_2 ds_3 < \infty. \tag{66}
\]

By Theorem 5.3, for \( g \in \overline{H}_1 \) and fixed \( \mathbf{p} \), we have the Fourier expansion

\[
g(s_2, s_3) \sim \sum_n c_{n, \mathbf{p}} E_{2, n, \mathbf{p}}^{(1)}(s_2) E_{3, n, \mathbf{p}}^{(1)}(s_3), \tag{67}
\]

where the Fourier coefficients are given by

\[
c_{n, \mathbf{p}} = \int_{a_1}^{a_2} \int_{a_1}^{a_2} \int_{a_1}^{a_2} \frac{s_3 - s_2}{\omega(s_2)\omega(s_3)} g(s_2, s_3) E_{2, n, \mathbf{p}}^{(1)}(s_2) E_{3, n, \mathbf{p}}^{(1)}(s_3) ds_2 ds_3. \tag{68}
\]

We are now ready to solve the Dirichlet problem \( \Delta u = 0 \) in \( D_1, u = e \) on \( \partial D_1 \) for \( D_1 \) given in (63). We will interpret the equation \( u = e \) on \( \partial D_1 \) in the following weak sense: if \( u \) and \( e \) are expressed in terms of 5-cyclide coordinates \( s_1, s_2, s_3, c_2, s_3 \), respectively, then \( u \) evaluated at \( s_1 \in (d_1, a_1) \) converges to \( e \) in the Hilbert space \( \overline{H} \) as \( s_1 \to d_1 \).

Theorem 6.3. Consider the region \( D_1 \) defined by (63) for some fixed \( d_1 \in (a_0, a_1) \). Let \( e \) be a function defined on its boundary \( \partial D_1 \) of parity \( \mathbf{p} \in [0, 1]^3 \), and let \( g(s_2, s_3) \) be the representation of \( f(x, y, z) := (x^2 + y^2 + z^2 + 1)^{1/2} e(x, y, z) \tag{69} \)

in 5-cyclide coordinates for \( (x, y, z) \in \partial D_1 \) with \( x, y, z > 0 \). Suppose \( g \in \overline{H}_1 \) and expand \( g \) in the series (67). Then the function \( u(x, y, z) \) given by

\[
u(x, y, z) = \sum_n c_{n, \mathbf{p}} E_{1, n, \mathbf{p}}^{(1)}(d_1) G_{n, \mathbf{p}}^{(1)}(x, y, z) \tag{70}
\]

is harmonic in \( D_1 \) and assumes the values \( e \) on the boundary of \( D_1 \) in the weak sense.

Proof. Let \( d_1 < d < a_1 \) and \( s_1 \in [d, a_1] \). Using Theorems 5.4, 5.6, we estimate

\[
\left| \frac{E_{1, n, \mathbf{p}}^{(1)}(s_1)}{E_{1, n, \mathbf{p}}^{(1)}(d_1)} E_{2, n, \mathbf{p}}^{(1)}(s_2) E_{3, n, \mathbf{p}}^{(1)}(s_3) \right| \leq |c_{n, \mathbf{p}}| C r^{n_2^2 + n_3^2} B(n_2^2 + n_3^2 + 1),
\]

where the constants \( B, C > 0 \) and \( r \in (0, 1) \) are independent of \( n \) and \( s_1 \in [d, a_1], s_2 \in [a_1, a_2], s_3 \in [a_2, a_3] \). Since \( c_{n, \mathbf{p}} \) is a bounded double sequence, this proves that the series in (70) is absolutely and uniformly convergent on compact subsets of \( D_1 \). Consequently, by Lemma 6.1, \( u(x, y, z) \) is harmonic in \( D_1 \). If we consider \( u \) for fixed \( s_1 \in (d_1, a_1) \) and compute the norm \( \|u - e\| \) in the Hilbert space \( \overline{H}_1 \) by the Parseval equality, we obtain

\[
\|u - e\|^2 \leq \sum_n |c_{n, \mathbf{p}}|^2 \left( 1 - \frac{E_{1, n, \mathbf{p}}^{(1)}(s_1)}{E_{1, n, \mathbf{p}}^{(1)}(d_1)} \right)^2.
\]

It is easy to see that the right-hand side converges to 0 as \( s_1 \to d_1 \). Taking into account that \( e \) and \( u \) have the same parity, it follows that \( u \) assumes the boundary values \( e \) in the weak sense. \( \Box \)

If \( e \) is a function on \( \partial D_1 \) without parity, we write the function \( f \) from (69) as a sum of eight functions

\[
f = \sum_p f_p.
\]
where \( f_p \) is of parity \( p \). Then the solution of the corresponding Dirichlet problem is given by

\[
u(x, y, z) = \sum_{n, p} \frac{c_{n, p}}{E_{1, n, p}(d_1)} G_{n, p}(x, y, z),
\]

where

\[
c_{n, p} = \int_{a_1}^{a_2} \int_{a_2}^{a_3} \frac{s_3 - s_2}{\omega(s_2)\omega(s_3)} g_p(s_2, s_3) E_{2, n, p}(s_2) E_{3, n, p}(s_3) \, ds_2 \, ds_3
\]

and \( g_p(s_2, s_3) \) is the representation of \( f_p \) in 5-cyclide coordinates.

We may write the coefficient \( c_{n, p} \) as an integral over the surface \( \partial D_1 \) itself. The surface element is \( dS = h_2 h_3 \, ds_2 \, ds_3 \) with the scale factors \( h_2, h_3 \) given in (24) and (25). Using

\[
\frac{h_2 h_3}{h_1} = \frac{1}{4}(x^2 + y^2 + z^2 + 1) - \frac{\omega(s_1)}{\omega(s_2)\omega(s_3)} (s_3 - s_2),
\]

we obtain from (72)

\[
c_{n, p} = \frac{1}{2\omega(d_1)E_{1, n, p}(d_1)} \int_{\partial D_1} \frac{e}{h_1} G_{n, p} dS,
\]

where

\[
h_1^2 = \frac{1}{16} \left( \frac{(x^2 + y^2 + z^2 - 1)^2}{(d_1 - a_0)^2} + \frac{4x^2}{(d_1 - a_1)^2} + \frac{4y^2}{(d_1 - a_2)^2} + \frac{4z^2}{(d_1 - a_3)^2} \right).
\]

VII. SECOND TWO-PARAMETER STEURM-LIOUVILLE PROBLEM

We treat the two-parameter eigenvalue problem that appears when we wish to solve the Dirichlet problem in asymmetric ring cyclides. It is quite similar to the one considered in Sec. V; however, there are also some interesting differences. Consider Eq. (26) on the intervals \((a_0, a_1)\) and \((a_2, a_3)\). We obtain two Sturm-Liouville equations involving two parameters

\[
(\omega(s_1)w_1')' - \frac{1}{\omega(s_1)} \left( \frac{3}{16} s_1^2 + \lambda_1 s_1 + \lambda_2 \right) w_1 = 0, \quad a_0 < s_1 < a_1,
\]

\[
(\omega(s_3)w_3')' - \frac{1}{\omega(s_3)} \left( \frac{3}{16} s_3^2 + \lambda_1 s_3 + \lambda_2 \right) w_3 = 0, \quad a_2 < s_3 < a_3.
\]

We again simplify by substituting \( t_j = \Omega(s_j), u_j(t_j) = w_j(s_j) \). Then (74) and (75) become

\[
u_1'' - \left( \frac{3}{16} \phi(t_1)^2 + \lambda_1 \phi(t_1) + \lambda_2 \right) u_1 = 0, \quad b_0 \leq t_1 \leq b_1,
\]

\[
u_3'' - \left( \frac{3}{16} \phi(t_3)^2 + \lambda_1 \phi(t_3) + \lambda_2 \right) u_3 = 0, \quad b_2 \leq t_3 \leq b_3.
\]

We add boundary conditions

\[
u_1'(b_0) = u_1'(b_1) = u_3'(b_2) = u_3'(b_3) = 0.
\]

Differential equations (76) and (77) together with boundary conditions (78) pose a two-parameter Sturm-Liouville eigenvalue problem. In contrast to Sec. V, we now have a uniformly right-definite problem:

\[
- \left| \begin{array}{c} \phi(t_1) \\ \phi(t_3) \end{array} \right| \geq \phi(t_3) - \phi(t_1) \geq a_2 - a_1 > 0 \quad \text{for } b_0 \leq t_1 \leq b_1 \leq t_3 \leq b_3.
\]
We again have Klein’s oscillation theorem.

**Theorem 7.1.** For every \( n = (n_1, n_3) \in \mathbb{N}_0^2 \), there exists a uniquely determined eigenvalue \((\lambda_{1,n}, \lambda_{2,n}) \in \mathbb{R}^2\) admitting an eigenfunction \(u_1\) with exactly \(n_1\) zeros in \((b_0, b_1)\) and an eigenfunction \(u_3\) with exactly \(n_3\) zeros in \((b_2, b_3)\).

We state a result on the distribution of eigenvalues.

**Theorem 7.2.** There are constants \(A_1, A_2, A_3 > 0\) such that, for all \( n \in \mathbb{N}_0^2 \),

\[
-A_1(n_1^2 + 1) \leq \lambda_{1,n} \leq A_2(n_1^2 + 1),
\]

\[
|\lambda_{2,n}| \leq A_3(n_1^2 + n_3^2 + 1).
\]

**Proof.** We abbreviate \(\lambda_j = \lambda_{j,n}\). Arguing as in the proof of Theorem 5.2, there are \(t_1 \in [b_0, b_1]\) and \(t_3 \in [b_2, b_3]\) such that

\[
\frac{3}{16} \left( (\phi(t_1))^2 + \lambda_1 \phi(t_1) + \lambda_2 \right) = -\frac{\pi^2 n_1^2}{(b_1 - b_0)^2},
\]

\[
\frac{3}{16} \left( (\phi(t_3))^2 + \lambda_1 \phi(t_3) + \lambda_2 \right) = -\frac{\pi^2 n_3^2}{(b_3 - b_2)^2}.
\]

By subtracting (81) from (82), we obtain

\[
\frac{3}{16} \left( (\phi(t_3))^2 - (\phi(t_1))^2 \right) + \lambda_3 (\phi(t_3) - \phi(t_1)) = -\frac{\pi^2 n_1^2}{(b_1 - b_0)^2} - \frac{\pi^2 n_3^2}{(b_3 - b_2)^2},
\]

which implies (79). Now (80) follows from (79) and (81).

Let \(u_{1,n}\) and \(u_{3,n}\) denote eigenfunctions corresponding to the eigenvalue \((\lambda_{1,n}, \lambda_{2,n})\). The system of products \(u_{1,n}(t_1)u_{3,n}(t_3), n \in \mathbb{N}_0^2\), is orthogonal in the Hilbert space \(H_2\) consisting of measurable functions \(f : (b_0, b_1) \times (b_2, b_3) \to \mathbb{C}\) satisfying

\[
\int_{b_2}^{b_3} \int_{b_0}^{b_1} (\phi(t_3) - \phi(t_1)) |f(t_1, t_3)|^2 \, dt_1 \, dt_3 < \infty
\]

with inner product

\[
\int_{b_2}^{b_3} \int_{b_0}^{b_1} (\phi(t_3) - \phi(t_1)) f(t_1, t_3) g(t_1, t_3) \, dt_1 \, dt_3.
\]

We normalize the eigenfunctions so that

\[
\int_{b_2}^{b_3} \int_{b_0}^{b_1} (\phi(t_3) - \phi(t_1)) \left[ u_{1,n}(t_1) \right]^2 \left[ u_{3,n}(t_3) \right]^2 \, dt_1 \, dt_3 = 1.
\]

We have the following completeness theorem.

**Theorem 7.3.** The double sequence of functions

\[
u_{1,n}(t_1)u_{3,n}(t_3), \quad n \in \mathbb{N}_0^2,
\]

forms an orthonormal basis in the Hilbert space \(H_2\).

The normalization (83) leads to a bound on the values of eigenfunctions. Since we have uniform right-definiteness, the proof is simpler than the proof of Theorem 5.4.
Theorem 7.4. There is a constant $B > 0$ such that, for all $n \in N^2_0$ and all $t_1 \in [b_0, b_1]$, $t_3 \in [b_2, b_3]$,

$$|u_{1,n}(t_1)u_{3,n}(t_3)| \leq B(n_1^2 + n_3^2 + 1).$$

Let $u_{2,n}$ be the solution of

$$u''_n + \left( \frac{3}{16}(\phi(t_2))^2 + \lambda_{1,n}\phi(t_2) + \lambda_{2,n} \right) u_2 = 0, \quad b_1 \leq t_2 \leq b_2$$

determined by initial conditions

$$u_2(b_1) = 1, \quad u'_2(b_1) = 0.$$

Theorem 7.5. We have $u_{2,n}(t_2) > 0$ for all $t_2 \in [b_1, b_2]$. If $b_1 < c_1 < c_2 < b_2$, then there are constants $C > 0$ and $0 < r < 1$ such that, for all $n \in N^2_0$ and $t_2 \in [b_1, c_1]$,

$$\left| \frac{u_{2,n}(t_2)}{u_{2,n}(c_2)} \right| \leq Cr^{n_1+n_3}.$$

Proof. We abbreviate $u_2 = u_{2,n}$ and $\lambda_j = \lambda_{j,n}$. We write (84) in the form

$$u''_2 + Q(\phi(t_2))u_2 = 0, \quad t_2 \in [b_1, b_2],$$

where $Q$ is given by (50). According to (81) and (82), there are $s_1 \in (a_0, a_1)$ and $s_3 \in (a_2, a_3)$ such that

$$Q(s_1) = \frac{-\pi^2 n_1^2}{(b_1 - b_0)^2}, \quad Q(s_3) = \frac{-\pi^2 n_3^2}{(b_3 - b_2)^2}.$$

If $s \in [s_1, s_3]$, then $Q(s) \leq L(s)$, where $L(s)$ is the linear function with $L(s_j) = Q(s_j), j = 1, 3$. It follows that

$$Q(\phi(t_2)) \leq -C(n_1 + n_3)^2 \quad \text{for} \quad t_2 \in [c_1, c_2].$$

We use a modification of Lemma 5.7 to complete the proof. □

The results of this section remain valid for other boundary conditions. This time we will need sixteen sets of boundary conditions labeled by $p = (p_0, p_1, p_2, p_3) \in \{0, 1\}^4$. These boundary conditions are

$$\begin{align*}
u'_1(b_0) &= 0 \quad \text{if} \quad p_0 = 0, \quad u_1(b_0) = 0 \quad \text{if} \quad p_0 = 1, \\
u'_1(b_1) &= 0 \quad \text{if} \quad p_1 = 0, \quad u_1(b_1) = 0 \quad \text{if} \quad p_1 = 1, \\
u'_2(b_2) &= 0 \quad \text{if} \quad p_2 = 0, \quad u_2(b_2) = 0 \quad \text{if} \quad p_2 = 1, \\
u'_3(b_3) &= 0 \quad \text{if} \quad p_3 = 0, \quad u_3(b_3) = 0 \quad \text{if} \quad p_3 = 1.
\end{align*}$$

The initial conditions for $u_2$ are

$$u_2(b_1) = 1, \quad u'_2(b_1) = 0 \quad \text{if} \quad p_1 = 0, \quad u_2(b_1) = 0, \quad u'_2(b_1) = 1 \quad \text{if} \quad p_1 = 1.$$

We denote the corresponding eigenvalues by $(\lambda^{(2)}_{1,n,p}, \lambda^{(2)}_{2,n,p})$. The eigenfunctions will be denoted by $E^{(2)}_{i,n,p}(s_i) = u_{i,n}(t_i), i = 1, 2, 3$.

Summarizing, for $i = 1, 2, 3$, $E^{(2)}_{i,n,p}$ is a solution of (26) on $(a_{i-1}, a_i)$ with $(\lambda_1, \lambda_2) = (\lambda^{(2)}_{1,n,p}, \lambda^{(2)}_{2,n,p})$. The solution $E^{(2)}_{1,n,p}(s_1)$ has exponent $\frac{1}{2}p_0$ at $a_0$, exponent $\frac{1}{2}p_1$ at $a_1$, and it has $n_1$ zeros in $(a_0, a_1)$. The solution $E^{(2)}_{2,n,p}(s_2)$ has exponent $\frac{1}{2}p_1$ at $a_1$, and it has no zeros in $(a_1, a_2)$. The solution $E^{(2)}_{3,n,p}(s_3)$ has exponent $\frac{1}{2}p_2$ at $a_2$, exponent $\frac{1}{2}p_3$ at $a_3$, and it has $n_3$ zeros in $(a_2, a_3)$.
FIG. 5. Coordinate surfaces $s_2, s_3 = \text{const}$ for $a_i = i$.

VIII. SECOND DIRICHLET PROBLEM

Consider the coordinate surface (21) for fixed $s = d_2 \in (a_1, a_2)$. See Figures 5(a)–5(c) for a graphical depiction of the shape of this surface. If $(x', y', z') \in S_2$ then the ray $(x, y, z) = t(x', y', z'), t > 0$, is tangent to the surface if and only if $(x', y', z')$ is on the surface. If $(x', y', z')$ is in the elliptical cone

$$\frac{4x'^2}{d_2 - a_1} + \frac{4y'^2}{d_2 - a_2} + \frac{4z'^2}{d_2 - a_3} > 0,$$

then the ray does not intersect the surface. Otherwise we have two intersections $t = t_1, t_2$ and $t_1 t_2 = 1$. It follows from these considerations that $s_2 = d_2$ describes a connected surface of genus one. The region interior to this surface is

$$D_2 = \{(x, y, z) : s_2 < d_2\},$$

or, equivalently,

$$D_2 = \{(x, y, z) : \frac{d_2 - a_0}{d_2 - a_0} + \frac{4x^2}{d_2 - a_1} + \frac{4y^2}{d_2 - a_2} + \frac{4z^2}{d_2 - a_3} < 0\}.$$

In this section, we solve the Dirichlet problem for harmonic functions in $D_2$ by the method of separation of variables.

Let $p = (p_0, p_1, p_2, p_3) \in \{0, 1\}^4$ and $n = (n_1, n_3) \in \mathbb{N}_0^2$. Using the functions $E^{(2)}_{i,n,p}$ introduced in Sec. VII, we define the internal 5-cyclidic harmonic of the second kind

$$G_{n,p}^{(2)}(x, y, z) = (x^2 + y^2 + z^2 + 1)^{-1/2} E_{1,n,p}^{(2)}(s_1) E_{2,n,p}^{(2)}(s_2) E_{3,n,p}^{(2)}(s_3)$$

for $x, y, z \in B_1(0)$ with $x, y, z \geq 0$. We extend the function

$$(x^2 + y^2 + z^2 + 1)^{1/2} G_{n,p}^{(2)}(x, y, z)$$

to $\mathbb{R}^3$ as a function of parity $p$. We call a function $f$ of parity $p = (p_0, p_1, p_2, p_3)$ if

$$f(s_i(x, y, z)) = (-1)^{p_i} f(x, y, z), \quad \text{for } i = 0, 1, 2, 3,$$

using inversion (29) and reflections (30).

We omit the proof of the following lemma which is similar to the proof of Lemma 6.1.
Lemma 8.1. The function $G_{n,p}^{(2)}$ is harmonic at all points $(x, y, z) \in \mathbb{R}^3$ at which $s_2 \neq a_2$; see (34).

Note that $s_2 < d_2 < a_2$ in $D_2$. Therefore, $G_{n,p}^{(2)}$ is harmonic in an open set containing the closure of $D_2$. Geometrically speaking, the set $s_2 = a_2$ consists of the part of the plane $y = 0$ “outside” the two closed curves in Figure 3. The asymmetric ring cyclides $D_2$ passes through the $y = 0$ plane inside those two closed curves.

Substituting $t_j = \Omega(s_j), j = 1, 3$, the Hilbert space $H_2$ from Sec. VII transforms to the Hilbert space $\tilde{H}_2$ consisting of measurable functions $g : (a_0, a_1) \times (a_2, a_3) \to \mathbb{C}$ for which

$$
\|g\|^2 := \int_{a_1}^{a_2} \int_{a_0}^{a_1} |g(s_1, s_3)|^2 ds_1 ds_3 < \infty.
$$

By Theorem 7.3, for $g \in \tilde{H}_2$ and fixed $p$, we have the Fourier expansion

$$
g(s_1, s_3) \sim \sum_n c_{n,p} E_{1,n,p}^{(2)}(s_1) E_{3,n,p}^{(2)}(s_3),
$$

where the Fourier coefficients are given by

$$
c_{n,p} = \int_{a_1}^{a_2} \int_{a_0}^{a_1} \frac{s_3 - s_1}{\omega(s_1) \omega(s_3)} g(s_1, s_3) E_{2,n,p}^{(2)}(s_1) E_{3,n,p}^{(2)}(s_3) ds_1 ds_3.
$$

Theorem 8.2. Consider the region $D_2$ defined by (88) for some fixed $d_2 \in (a_1, a_2)$. Let $e$ be a function defined on its boundary $\partial D_2$, and set

$$
f(x, y, z) := (x^2 + y^2 + z^2 + 1)^{1/2} e(x, y, z).
$$

Suppose that $f$ has parity $p \in \{0, 1\}$, and its representation $g(s_1, s_3)$ in 5-cyclide coordinates is in $\tilde{H}_2$. Expand $g$ in the series (92). Then the function $u(x, y, z)$ given by

$$
u(x, y, z) = \sum_n c_{n,p} E_{2,n,p}^{(2)}(d_2) G_{n,p}^{(2)}(x, y, z)
$$

is harmonic in $D_2$ and assumes the values $e$ on the boundary of $D_2$ in the weak sense.

Proof. The proof is similar to the proof of Theorem 6.3. It uses Theorems 7.4 and 7.5 to show that the series in (94) is absolutely and uniformly convergent on compact subsets of $D_2$. Consequently, by Lemma 8.1, $u(x, y, z)$ is harmonic in $D_2$. If we consider $u$ for fixed $s_2 \in (a_1, d_2)$ and compute the norm $\|u - e\|$ in the Hilbert space $\tilde{H}_2$, we obtain

$$
\|u - e\|^2 \leq \sum_n |c_{n,p}|^2 \left(1 - \frac{E_{2,n,p}^{(2)}(s_2)}{E_{2,n,p}^{(2)}(d_2)}\right)^2.
$$

The right-hand side converges to 0 as $s_2 \to d_2$. Hence $u$ assumes the boundary values $e$ in the weak sense.

If $f$ is a function without parity, we write $f$ as a sum of 16 functions

$$
f = \sum_{p \in \{0, 1\}^4} f_p,
$$

where $f_p$ is of parity $p$. Then the solution of the corresponding Dirichlet problem is given by

$$
u(x, y, z) = \sum_n \frac{c_{n,p}}{E_{2,n,p}^{(2)}(d_2)} G_{n,p}^{(2)}(x, y, z),
$$

where

$$
c_{n,p} = \int_{a_1}^{a_2} \int_{a_0}^{a_1} \frac{s_3 - s_1}{\omega(s_1) \omega(s_3)} g_p(s_1, s_3) E_{1,n,p}^{(2)}(s_1) E_{3,n,p}^{(2)}(s_3) ds_1 ds_3
$$

and set

$$
E_{p}^{(2)}(x_0) = \int_{x_0}^{a_1} \frac{s_3 - s_1}{\omega(s_1) \omega(s_3)} g(s_1, s_3) ds_1 ds_3.
$$

Note that $E_{p}^{(2)}$ is a function without parity, we write $E_{p}^{(2)}$ as a sum of 16 functions

$$
E_{p}^{(2)} = \sum_{p \in \{0, 1\}^4} E_{p}^{(2)}.
$$

The proof is similar to the proof of Theorem 6.3. It uses Theorems 7.4 and 7.5 to show that the series in (94) is absolutely and uniformly convergent on compact subsets of $D_2$. Consequently, by Lemma 8.1, $u(x, y, z)$ is harmonic in $D_2$. If we consider $u$ for fixed $s_2 \in (a_1, d_2)$ and compute the norm $\|u - e\|$ in the Hilbert space $\tilde{H}_2$, we obtain

$$
\|u - e\|^2 \leq \sum_n |c_{n,p}|^2 \left(1 - \frac{E_{2,n,p}^{(2)}(s_2)}{E_{2,n,p}^{(2)}(d_2)}\right)^2.
$$

The right-hand side converges to 0 as $s_2 \to d_2$. Hence $u$ assumes the boundary values $e$ in the weak sense. □
and $g_p(s_1, s_2)$ is the representation of $f_p$ in 5-cyclide coordinates. We may also write $c_{n,p}$ as a surface integral

$$c_{n,p} = \frac{1}{4\omega(d_2)E_{2,n,p}^2(d_2)} \int_{\partial D} e^{s_i G_{n,p}^2} dS,$$

where

$$h_2^2 = \frac{1}{16} \left( \frac{x^2 + y^2 + z^2 - 1}{(d_2 - a_0)^2} + \frac{4x^2}{(d_2 - a_1)^2} + \frac{4y^2}{(d_2 - a_2)^2} + \frac{4z^2}{(d_2 - a_3)^2} \right).$$

**IX. THIRD TWO-PARAMETER STURM-LIOUVILLE PROBLEM**

If we write (26) on the intervals $(a_0, a_1)$ and $(a_1, a_2)$ in formally self-adjoint form, we obtain

$$(\omega(s_1)w_1')' - \frac{1}{\omega(s_1)} \left( \frac{3}{16} s_1^2 + \lambda_1 s_1 + \lambda_2 \right) w_1 = 0, \quad a_0 < s_1 < a_1,$$  

$$(\omega(s_2)w_2')' + \frac{1}{\omega(s_2)} \left( \frac{3}{16} s_2^2 + \lambda_1 s_2 + \lambda_2 \right) w_2 = 0, \quad a_1 < s_2 < a_2.$$  

We simplify the equations by substituting $t_j = \Omega(s_j)$, $u_j(t_j) = w_j(s_j)$, where $\Omega(s)$ is the elliptic integral (42). Then (98) and (99) become

$$u_1' - \left( \frac{3}{16} (\phi(t_1))^2 + \lambda_1 \phi(t_1) + \lambda_2 \right) u_1 = 0, \quad b_0 \leq t_1 \leq b_1,$$  

$$u_2' + \left( \frac{3}{16} (\phi(t_2))^2 + \lambda_1 \phi(t_2) + \lambda_2 \right) u_2 = 0, \quad b_1 \leq t_2 \leq b_2.$$  

Of course, this system is very similar to the one considered in Sec. V. Therefore, we will be brief. For a given $p = (p_0, p_1, p_2) \in [0, 1]^3$, we consider the boundary conditions

$$u_1'(b_0) = 0 \quad \text{if } p_0 = 0, \quad u_1(b_0) = 0 \quad \text{if } p_0 = 1,$$  

$$u_1'(b_1) = u_2'(b_1) = 0 \quad \text{if } p_1 = 0, \quad u_1(b_1) = u_2(b_1) = 0 \quad \text{if } p_1 = 1,$$  

$$u_2'(b_2) = 0 \quad \text{if } p_2 = 0, \quad u_2(b_2) = 0 \quad \text{if } p_2 = 1.$$  

The initial conditions for $u_3$ are

$$u_3(b_2) = 1 \quad \text{if } p_2 = 0, \quad u_3(b_2) = 0 \quad \text{if } p_2 = 1.$$  

We denote the corresponding eigenvalues by $(\lambda_{1,n,p}^{(3)}, \lambda_{2,n,p}^{(3)})$, where $n = (n_1, n_2) \in \mathbb{N}_0^2$. The eigenfunctions will be denoted by $E_{i,n,p}(s_i) = u_{i,n}(t_i)$, $i = 1, 2, 3$.

Summarizing, for $i = 1, 2, 3$, $E_{i,n,p}^{(3)}$ is a solution of (26) on $(a_{i-1}, a_i)$ with $(\lambda_1, \lambda_2) = (\lambda_{1,n,p}^{(3)}, \lambda_{2,n,p}^{(3)})$. The solution $E_{i,n,p}(s_1)$ has exponent $\frac{1}{2} p_0$ at $a_0$, exponent $\frac{1}{2} p_1$ at $a_1$, and it has $n_1$ zeros in $(a_0, a_1)$. The solution $E_{2,n,p}^{(3)}(s_2)$ has exponent $\frac{1}{2} p_1$ at $a_1$, exponent $\frac{1}{2} p_2$ at $a_2$, and it has $n_2$ zeros in $(a_1, a_2)$. The solution $E_{3,n,p}^{(3)}(s_3)$ has exponent $\frac{1}{2} p_2$ at $a_2$, and it has no zeros in $(a_2, a_3)$.

**X. THIRD DIRICHLET PROBLEM**

Consider the coordinate surface (21) for fixed $s = d_2 \in (a_2, a_3)$. See Figures 5(d)–5(f) for a graphical depiction of the shape of this surface. If $(x', y', z') \in S_2$, then the ray $(x, y, z) = t(x', y', z')$, $t > 0$, is tangent to the surface if and only if $(x', y', z')$ is on the surface. If $(x', y', z')$ is in the elliptical cone

$$\frac{4x^2}{d_2 - a_1} + \frac{4y^2}{d_2 - a_2} + \frac{4z^2}{d_2 - a_3} < 0,$$
then the ray intersects the surface twice at \( t = t_1, t_2 \) with \( t_1 t_2 = 1 \). Otherwise, there is no intersection. Therefore, the coordinate surface \( s_3 = d_3 \) consists of two disjoint closed asymmetric surfaces of genus one separated by the plane \( z = 0 \), and they are mirror images of each other under the reflection \( \sigma_3 \).

We consider the region inside the surface \( s_3 = d_3 \) with \( z > 0 \)
\[
D_3 = \{(x, y, z) : z > 0, s_3 < d_3\},
\]
or, equivalently,
\[
D_3 = \{(x, y, z) : z > 0, \frac{(x^2 + y^2 + z^2 - 1)^2}{d_3 - a_0} + \frac{4x^2}{d_3 - a_1} + \frac{4y^2}{d_3 - a_2} + \frac{4z^2}{d_3 - a_3} < 0\}.
\]

Next, we solve the Dirichlet problem for harmonic functions in \( D_3 \) by the method of separation of variables.

Let \( p = (p_0, p_1, p_2) \in \{0, 1\}^3 \) and \( n = (n_1, n_2) \in \mathbb{N}_0^2 \). Using the functions \( E_{i,n,p}^{(3)} \) introduced in Sec. IX, we define the internal 5-cyclidic harmonic of the third kind
\[
G_{n,p}^{(3)}(x, y, z) = (x^2 + y^2 + z^2 + 1)^{-1/2} E_{1,n,p}^{(3)}(s_1) E_{2,n,p}^{(3)}(s_2) E_{3,n,p}^{(3)}(s_3)
\]
for \((x, y, z) \in B_1(0)\) with \( x, y, z \geq 0 \). We extend the function
\[
(x^2 + y^2 + z^2 + 1)^{1/2} G_{n,p}^{(3)}(x, y, z)
\]
to the half-space \( \{(x, y, z) : z > 0\} \) as a function of parity \( p \). We call a function \( f \) of parity \( p = (p_0, p_1, p_2) \), if
\[
f(\sigma_i(x, y, z)) = (-1)^{p_i} f(x, y, z), \quad \text{for } i = 0, 1, 2
\]
using the inversion \( \sigma_0 \) and the reflections \( \sigma_1, \sigma_2 \). As before we have the following lemma.

**Lemma 10.1.** The function \( G_{n,p}^{(3)} \) is harmonic on \( \{(x, y, z) : z > 0\} \).

We have the Hilbert space \( \tilde{H}_3 \) consisting of measurable functions \( g : (a_0, a_1) \times (a_1, a_2) \to \mathbb{C} \) for which
\[
\|g\|^2 := \int_{a_1}^{a_2} \int_{a_0}^{a_1} \frac{s_2 - s_1}{\omega(s_1)\omega(s_2)} |g(s_1, s_2)|^2 ds_1 \, ds_2 < \infty.
\]

For \( g \in \tilde{H}_3 \) and fixed \( p \), we have the Fourier expansion
\[
g(s_1, s_2) \sim \sum_n c_{n,p} E_{1,n,p}^{(3)}(s_1) E_{2,n,p}^{(3)}(s_2),
\]
where the Fourier coefficients are given by
\[
c_{n,p} = \int_{a_1}^{a_2} \int_{a_0}^{a_1} \frac{s_2 - s_1}{\omega(s_1)\omega(s_2)} g(s_1, s_2) E_{1,n,p}^{(3)}(s_1) E_{2,n,p}^{(3)}(s_2) \, ds_1 \, ds_2.
\]

**Theorem 10.2.** Consider the region \( D_3 \) defined by (104) for some fixed \( d_3 \in (a_2, a_3) \). Let \( e \) be a function defined on its boundary \( \partial D_3 \), and set
\[
f(x, y, z) := (x^2 + y^2 + z^2 + 1)^{1/2} e(x, y, z).
\]
Suppose that \( f \) has parity \( p \in \{0, 1\}^3 \), and its representation \( g(s_1, s_2) \) in 5-cyclide coordinates is in \( \tilde{H}_3 \). Expand \( g \) in the series (108). Then the function \( u(x, y, z) \) given by
\[
u(x, y, z) = \sum_n c_{n,p} E_{3,n,p}(d_3) G_{n,p}^{(3)}(x, y, z)
\]
is harmonic in \( D_3 \) and assumes the values \( e \) on the boundary of \( D_3 \) in the weak sense.
If $f$ is a function without parity, we write $f$ as a sum of eight functions

$$f = \sum_{p \in \{0, 1\}} f_p,$$

where $f_p$ is of parity $p$. Then the solution of the corresponding Dirichlet problem is given by

$$u(x, y, z) = \sum_{n, p} c_{n,p} E^{(3)}_{3, n, p}(d_3) G^{(3)}_{n, p}(x, y, z),$$

where

$$c_{n,p} = \int_{a_0}^{s_2} \int_{a_1}^{s_1} \frac{s_2 - s_1}{\omega(s_1)\omega(s_2)} g_p(s_1, s_2) E^{(3)}_{1, n, p}(s_1) E^{(3)}_{2, n, p}(s_2) \, ds_1 \, ds_2,$$ (113)

and $g_p(s_1, s_2)$ is the representation of $f_p$ in 5-cyclide coordinates. Alternatively, we have

$$c_{n,p} = \frac{1}{2\omega(d_3)E^{(3)}_{3, n, p}(d_3)} \int_{\partial D} e^{G^{(3)}_{n, p}} \, dS,$$ (114)

where

$$h_3 = \frac{1}{16} \left( \frac{x^2 + y^2 + z^2 - 1}{(d_3 - a_0)^2} + \frac{4x^2}{(d_3 - a_1)^2} + \frac{4y^2}{(d_3 - a_2)^2} + \frac{4z^2}{(d_3 - a_3)^2} \right).$$

ACKNOWLEDGMENTS

This work was conducted while H. S. Cohl was a National Research Council Research Postdoctoral Associate in the Applied and Computational Mathematics Division at the National Institute of Standards and Technology, Gaithersburg, Maryland, USA.