The axes of random infinitesimal rotations and the propagation of orientation uncertainty

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Abstract

Perception systems can measure the orientation of a solid 3D object; however, their measurements will contain some uncertainties. In many robotic applications, it is important to propagate the orientation uncertainties of a rigid object onto the uncertainties of specific points on its surface. The orientation uncertainty can be reported as a 3×3 covariance matrix. We show that the off-diagonal elements of this matrix provide important clues about the angular uncertainties of points on the object’s surface. Specifically, large off-diagonal elements correspond to a highly concentrated distribution of axes of random infinitesimal rotations which causes large variability in the angular uncertainties of surface points. In particular, experimental data indicate that the ratio of maximum to minimum angular uncertainties can exceed three. In contrast, small off-diagonal elements correspond to a uniform distribution of axes which causes the angular uncertainty of all points on the object’s surface to be almost constant.

1. Introduction

In many robotic applications it is important to understand the uncertainty of specific points on the surface of a rigid body. For example, when planning a safe path of a solid object, the specific points are those that may come close to an obstacle; or when planning a good grip, the specific points are those that may come in contact with a gripper’s fingers. Typically, the calculation of these specific points is dependent on noisy orientation data obtained with a pose measuring system. We are interested in characterizing the propagation of the uncertainty of this noisy orientation data onto specific points on the surface of a rigid body.

Typically, in robotic applications the propagation of orientation uncertainty is defined as the propagation along different joints (encoders) on a kinematic chain, or in the context of dynamic control the propagation of orientation uncertainty is defined when the orientation at time k + 1 depends on noisy orientation measured at time k [1,2]. In contrast, here we are interested in the static configuration where a fixed orientation of a rigid body is repeatedly measured in the same experimental conditions. Mathematically, this can be modeled by studying the uncertainty of

\[ w_j = R_j u \]  

where \( R_j \) is a 3×3 rotation matrix representing the noisy orientation data obtained from a pose measuring system at the j-th measurement and \( u \) is a 3D vector representing a specific point on the surface of a rigid body. We are interested in characterizing how the propagated uncertainty changes among different points (\( u \)) on the surface of the rigid body.
For this paper, we will make some assumptions on the orientation $R_j$ and point $u$. In particular, we assume that the measured orientation can be broken down as

$$R_j = R_0 \Delta R_j$$

where $R_0$ is the true and usually unknown rotation and $\Delta R_j$ is a random infinitesimal rotation representing the uncertainty of the orientation information at the $j$-th measurement [3]. We will approximate the unknown true rotation $R_0$ with the average rotation $R_{\text{avg}}$ calculated from averaging many $R_j$ ($j = 1, 2, \ldots, N$) that are measured in the same experimental conditions as outlined in Guide to the Expression of Uncertainty (GUM) [4]. We should note, however, that there are different ways to calculate $R_{\text{avg}}$ [5].

From (2), we can see that the statistical properties of the measured rotations $R_j$ are defined by the characteristics of the random infinitesimal rotations $\Delta R_j$. In the axis-angle representation, an infinitesimal rotation is defined by a small angle of rotation $\rho_j$ such that $\sin \rho_j \approx \rho_j$ and $\cos \rho_j \approx 1$, regardless of the direction of the corresponding axis. This mathematical property of the axis-angle representation may tempt one to conclude that the spatial distribution of the axis of rotation is irrelevant and does not have any important practical implications [6]. However, in this paper we will show that the axis of a random infinitesimal rotation is important in understanding the propagation of uncertainty from the orientation information of a rigid body onto a specific point on its surface.

Another aspect of the uncertainty propagation deals with linear scaling: points that are further away from the center of rotation (and are not on the axis of rotation) will have larger linear uncertainty compared to points that are closer to the center of rotation. As a result, we will only deal with points $u$ that lie on the unit sphere. Therefore, $u$ can be parameterized by two angles $- \vartheta$ elevation and azimuth $\varphi$ such that $u = u(\vartheta, \varphi)$. Since rotation does not change the length of a vector, the rotated vectors $w_j$ can also be parameterized by two angles $- \lambda$ elevation and azimuth $\tau$ such that $w_j = w(\lambda_j, \tau_j)$. Thus, we are interested in propagating error from the measured orientation $R_j$ onto the pair of angles $\lambda_j$ and $\tau_j$. We systematically investigate how the angular uncertainty of the corresponding $w_j$ depends on the covariance matrix of the three parameters defining the random infinitesimal rotations $\Delta R_j$ and the vector $u$ defining a point on the surface of a rigid body. Analysis of experimental data show that, though it may appear paradoxical, the angular uncertainty of different points on the surface of a rotated rigid body may vary up to a factor of three – even for a perfectly symmetrical sphere. We will show that this variability is strongly correlated with the distribution of the axes of the random infinitesimal rotations. Specifically, the variability will be large unless the axes of the random infinitesimal rotations are distributed uniformly about the unit sphere. This distribution of the axes can be determined by inspecting the off-diagonal elements of the covariance matrix of the three parameters defining the random infinitesimal rotations $\Delta R_j$. For large off-diagonal elements, the eigenvector of the covariance matrix corresponding to the smallest eigenvalue represents the point on the unit sphere with the largest uncertainty, while vectors that are perpendicular to this eigenvector represent the points where the smallest angular uncertainty is observed. Coincidentally, these perpendicular vectors also correspond to the region with the highest concentration of axes from the random infinitesimal rotations.

We will organize this paper in the following manner: Section 2 formulates the problem more precisely and reviews the necessary equations, Section 3 describes the experimental setup for obtaining the measured orientations $R_j$ and subsequent data processing, Section 4 contains the results of propagating the uncertainty from the measured orientations $R_j$ onto $w_j$. Section 5 contains a discussion of the consequences of these results, and Section 6 contains concluding remarks.

2. Angular uncertainty of a unit vector

If the Cartesian coordinates of a point $w_j(x, y, z)$ have a Gaussian distribution with the constraint that $||w_j|| = 1$, then the probability of finding a vector $w_j$ on the unit sphere is described by the Fisher–Bingham–Kent (FBK) distribution [7]. FBK distributions are the core of directional statistics [8–10] and were first used in paleomagnetism to study the spatial distribution of magnetic properties in rocks [11]. In applications that are pertinent to perception in robotics, the FBK has recently been used to improve the alignment method which is based on calculating the normal and principal curvature directions on the surface of an object [12,13]. FBK can be used to find the probability of measuring a specific deviation angle $\mu_j$ between a unit vector $w_j$ and the vector of average direction $w_{\text{avg}}$. The distribution of these angles $\mu_j$ is described by two parameters: the angular uncertainty $\sigma$ and the eccentricity $\beta$. Here $\sigma$ (or equivalent concentration $\kappa = \sigma^{-2}$) describes the spread of $w_j$ around $w_{\text{avg}}$ such that a smaller $\sigma$ indicates a higher (tighter) concentration around $w_{\text{avg}}$, and $\beta$ describes the shape of the elliptical contour of a constant probability on the unit sphere: when $\beta = 0$ the contour is a circle centered at $w_{\text{avg}}$ and larger $\beta$ corresponds to a flatter contour.

While paleomagnetic data may be very noisy and their associated pdf may have large spread, modern pose measuring instruments can provide angular data with uncertainty $\sigma$ on the order of milliradians $\kappa$. For such small angular uncertainties $\sigma$, the angle $\mu_j$ will be small and the following approximation holds: $\cos \mu_j \approx 1 - 0.5\mu_j^2$ and $\sin \mu_j \approx \mu_j$. Using this approximation, the probability of measuring a specific angle $\mu$ can be calculated as the FBK

$$G_{\sigma, \beta}(\mu) d\mu = \mu \exp(-0.5\mu^2/\sigma^2) E_{\sigma, \beta}(\mu) d\mu$$

where $E_{\sigma, \beta}(\mu)$ is the Kent correction to the Fisher distribution due to the nonzero parameter $\beta$ along the azimuth angle $\eta$ around $w_{\text{avg}}$

$$E_{\sigma, \beta}(\mu) = \frac{1}{2\pi} \int_0^{2\pi} \sqrt{(1-2\beta\sigma^2)(1+2\beta\sigma^2)} \exp(\beta\mu^2 \cos 2\eta) d\eta.$$
Parameters $\beta$ and $\sigma$ (or $\beta$ and $\kappa$) can be calculated from the zero centered covariance matrix for a set of vectors $\{w_j\}$ as in [7]. This probability measure is used to process the noisy orientation data acquired in the experiment described in Section 3.

From the many possible parameterizations of a rotation matrix $R$ [14], we chose the axis-angle representation $(v, \omega)$ which obeys the Rodrigues formulation with assumed right hand convention

$$R = R(v, \omega) = \cos(\omega)I + \sin(\omega)[v]_x + (1 - \cos(\omega))v^Tv,$$

where $I$ is the identity matrix, $v = (v^x, v^y, v^z)^T$ is a unit column vector, and $v^T$ is its transpose. We should note that this axis-angle representation for a rotation is equivalent to the quaternion representation [15]

$$g(\omega, v) = \left(\cos\left(\frac{\omega}{2}\right), \sin\left(\frac{\omega}{2}\right)v\right).$$

Since $v$ is a unit vector, it can be parameterized by a pair of elevation and azimuth angles $(\zeta, \psi)$ as

$$v(\zeta, \psi) = \left[\cos\zeta \cos\psi, \cos\zeta \sin\psi, \sin\zeta\right]^T.$$

In the remaining part of the paper, we will parameterize a unit vector by the elevation and azimuth angles as in (7). Thus, a unit vector will be represented by a point on the surface of a unit sphere in 3D parameterized by two angles as in (7). As a result, a given rotation matrix $R$ may be parameterized by three angles $(\zeta, \psi, \omega)$ such that $R'(\zeta, \psi, \omega) = R(v(\zeta, \psi), \omega)$.

We are interested in understanding how the noise in the measured orientations $R_j$ propagates to a specific point. In this paper, the infinitesimal rotation $\Delta R_j$ represents the noise in the measured rotation $R_j$. From (2), the noise can be represented as

$$\Delta R_j(a_j, \rho_j) = R_{avg} R_j,$$

since the inverse of the rotation matrix $R^{-1} = R^T$, and $a_j$ and $\rho_j$ are the respective axis and angle. Here the average rotation matrix

$$R_{avg} = R(\zeta_{avg}, \psi_{avg}, \omega_{avg})$$

where $(\zeta_{avg}, \psi_{avg}, \omega_{avg})$ are the average of $(\zeta_j, \psi_j, \omega_j)$ for $j = 1, \ldots, N$.

For the infinitesimal rotation in axis-angle representation, $\sin(\rho_j) \approx \rho_j$ and $\cos(\rho_j) \approx 1$. Then, the linearized version of the Rodrigues formulation applied to $\Delta R_j$ takes the form

$$\Delta R_j(a_j, \rho_j) \approx I_{3 \times 3} + \rho_j \begin{bmatrix} 0 & -a_j^y & a_j^y \\ a_j^x & 0 & -a_j^z \\ -a_j^y & a_j^z & 0 \end{bmatrix}$$

$$= I_{3 \times 3} + \begin{bmatrix} 0 & -q_j^x & q_j^y \\ q_j^x & 0 & -q_j^z \\ -q_j^y & q_j^z & 0 \end{bmatrix}$$

where

$$q_j = \rho_j a_j.$$

Note that the unit quaternion $g_j$ representing the approximated rotation $\Delta R_j$ can be written as

$$g_j = \left(\sqrt{1 - \frac{\rho_j}{2}}; \frac{1}{2}q_j\right).$$

The covariance matrix $\text{cov}(q)_{3 \times 3}$ of the orientation data can be calculated as

$$\text{cov}(q) = \frac{1}{N} [q_1 \ldots q_n][q_1 \ldots q_n]^T,$$

where $q_j = q_j - q_{avg}$. We note that any rotation has the strongest effect on vectors perpendicular to the axis of rotation. Therefore, we may expect that the eigenvector of the covariance matrix $\text{cov}(q)$ corresponding to the smallest eigenvalue will be pointing to the location on the surface of a 3D rigid body where the angular uncertainty $\sigma$ is the largest.

Columns of every rotation matrix form three mutually orthogonal unit vectors, i.e. $x_j = R_j(:, 1), y_j = R_j(:, 2)$ and $z_j = R_j(:, 3)$. For small noise, i.e., small standard deviations of $(\zeta_j, \psi_j, \omega)$, $R_{avg} \approx R_{avg}(, 1), \ Y_{avg} \approx R_{avg}(, 2)$ and $z_{avg} \approx R_{avg}(, 3)$. As a result, the deviation angles of the $j$-th frame axes $(x_j, y_j, z_j)$ from the averaged axes $(x_{avg}, y_{avg}, z_{avg})$ can be defined as

$$\angle \mu_j^x = \cos(x_j \cdot x_{avg}), \ \angle \mu_j^y = \cos(y_j \cdot y_{avg}), \ \angle \mu_j^z = \cos(z_j \cdot z_{avg}).$$

where $\cdot$ is the dot product of two vectors. It follows immediately from (12) that the diagonal elements of $\Delta R_j$ are cosines of the corresponding deviation angles

$$\Delta R_j(1, 1) = \cos(\angle \mu_1^x), \ \Delta R_j(2, 2) = \cos(\angle \mu_1^y), \ \Delta R_j(3, 3) = \cos(\angle \mu_1^z).$$

Since the trace of every rotation matrix, including the infinitesimal rotation matrix $\Delta R_j(a_j, \rho_j)$, determines the angle of rotation, we see that

$$\cos(\rho_j) = \cos(\angle \mu_j^x) + \cos(\angle \mu_j^y) + \cos(\angle \mu_j^z) - 1/2.$$

Therefore, the distribution of the angles of rotation $\rho_j$ is dependent on the distribution of the angles $\angle \mu_j^x, \angle \mu_j^y, \angle \mu_j^z$ each of which follows the Flick distribution.

3. Experiment and data processing

A commercial off-the-shelf optical pose measuring system was used to collect the orientation data. A solid object was mounted on a stand in a given orientation and several small spherical targets were attached to the object. The pose measuring system acquired the spatial locations of the targets to determine the orientation matrix $R_j$ of the object. The system outputted the roll, yaw, and pitch angles as well as the four components of the unit quaternion representation of $R_j$. From the quaternion representation, the axis and angle representation of $R_j(a_j(\zeta_j, \psi_j), \omega_j)$ can be directly calculated using (6) and (5).

The pose measuring system allowed a user to change the settings for the data acquisition such as the value of the detection threshold or the duration of the integration.
of the input signal. Different settings yielded different levels of noise in the collected data. For example, higher threshold and longer integration time resulted in cleaner output data while a lower threshold and shorter integration time caused more noise in the recorded data. For this experiment, the object was set at a given pose and three datasets were collected for high (A), low (B) and intermediate (C) instrument noise levels. Each dataset contained more than 50,000 repeated measurements.

For each recorded dataset, the average angles \( (\hat{\gamma}_\text{avg}, \hat{\psi}_\text{avg}; \hat{\omega}_\text{avg}) \) were calculated and histograms of the deviations \( \Delta_j = \hat{\gamma}_j - \gamma_\text{avg} \) were created for each \( \gamma_j = (\hat{\gamma}_j, \hat{\psi}_j, \hat{\omega}_j) \) (Fig. 1).

Once the average orientation \( \mathbf{R}_\text{avg} \) from (9) was computed, a histogram for each of the deviation angles \( \langle \hat{\mu}_0^r, \hat{\mu}_1^r, \hat{\mu}_2^r \rangle \) from (12) was calculated (Fig. 2). For each measured rotation \( \mathbf{R}_j \), the error was computed as the infinitesimal rotational matrix \( \Delta \mathbf{R} = \mathbf{R}_j \mathbf{R}_\text{avg}^{-1} \) using (8). For each dataset \( \{\Delta \mathbf{R}_j\} \) collected with a given instrument setting, a histogram of the corresponding infinitesimal rotation angles \( \rho_j \) was created (Fig. 3). In addition, for each dataset \( \{\Delta \mathbf{R}_j\} \) the histogram of the axes \( \mathbf{a}_j \) on the unit sphere was created (Fig. 4a and 4c). Next, the distribution of the angular uncertainty \( \sigma \) from (3) on the unit sphere was calculated in the following manner. For every pair of elevation and azimuth angles \( \langle \hat{\vartheta}_m, \hat{\varphi}_m \rangle \) on the unit sphere, a vector \( \mathbf{u}(\hat{\vartheta}_m, \hat{\varphi}_m) \) was built using (7). Then, for each acquired dataset of rotations \( \{\mathbf{R}_j\} \), a set of \( N \) rotated vectors \( \{\mathbf{w}_j\} \) was created using (1) for each starting vector \( \mathbf{u}(\hat{\vartheta}_m, \hat{\varphi}_m) \). This process yielded a distribution of the angular uncertainty \( \sigma(\hat{\vartheta}_m, \hat{\varphi}_m) \) calculated from a uniform distribution of the axes from random infinitesimal rotations. For each dataset of original random infinitesimal rotations \( \{\Delta \mathbf{R}_j\} \) and corresponding dataset of modified rotations \( \{\Delta \mathbf{R}_j\} \), a pair of covariant matrices \( \text{cov}(\mathbf{q}) \) and \( \text{cov}(\mathbf{q}) \) were calculated using (11).

4. Results

Fig. 1 shows the histograms of angles which parameterize the rotations \( \mathbf{R}(\hat{\gamma}_j, \hat{\psi}_j, \hat{\omega}_j) \), acquired for three different noise levels A–C at a fixed object orientation. Fig. 2 shows the corresponding histograms of the deviation angles \( \langle \hat{\mu}_0^r, \hat{\mu}_1^r, \hat{\mu}_2^r \rangle \) calculated from the same datasets which were used to generate Fig. 1. In Fig. 2, the symbols represent the experimental data and the solid lines represent the modified Fisher distributions \( G_{\sigma, \beta} \) given by (3). Parameters \( \sigma \) and \( \beta \) are calculated from the experimental data and the Kent correction \( E_{\sigma, \beta}(\mu) \) was calculated for each deviation angle \( \mu \) by numerically integrating (4). Fig. 3 displays the histogram of the angles \( \rho_j \) defining the infinitesimal rotational corrections \( \Delta \mathbf{R}(\mathbf{a}_j, \rho_j) \) in the axis-angle representation. In our calculations, we assumed that angle \( \rho_j \) is always positive but the axis \( \mathbf{a}_j \) can flip directions so that the right hand convention of the coordinate system is preserved. This assumption is reasonable due to the property of the axis-angle representation: \( \Delta \mathbf{R}(\mathbf{a}, -\rho) = \Delta \mathbf{R}(\mathbf{a}, \rho) \). Fig. 4a and c shows the distribution of the axes \( \mathbf{a}_j \) of random infinitesimal rotations \( \Delta \mathbf{R}_j \) on the unit sphere.

Fig. 4b and d shows the dependence of the angular uncertainty \( \sigma(\hat{\vartheta}, \hat{\varphi}) \) on the location of the point \( \mathbf{u}(\hat{\vartheta}, \hat{\varphi}) \) on the unit sphere. The straight line piercing the sphere surface in Fig. 4a–d is parallel to the direction of the eigenvector corresponding to the smallest eigenvalue of the covariance matrix \( \text{cov}(\mathbf{q}) \). The line defines the two poles and the equator on the unit sphere. The white mark on the surface of each colored sphere indicates one of the
two locations \( \pm u_{\text{min}} \) where the angular uncertainty \( \sigma_{\text{min}} \) is the smallest. The angular uncertainty \( \sigma \) along the equator does not deviate much from \( \sigma_{\text{min}} \): the largest relative deviation is less than 1.5%. The same maximum value of uncertainty \( r_{\text{max}} \) is located at both poles \( \pm u_{\text{max}} \). For all three instrument settings A–C, \( r_{\text{max}}/r_{\text{min}} \approx 25 \). For the original random rotations \( \{ R \} \), Fig. 5a shows the uniform distribution of axes \( \mathbf{a}_j \) of the modified random rotation \( \{ \Delta \mathbf{R} \} \) obtained from data acquired for instrument setting A and

\[ \Delta \mathbf{R}(\mathbf{a}, \rho) \]

Fig. 5b displays the corresponding dependence of the angular uncertainty \( \hat{\sigma}(\vartheta, \phi) \) on \( \mathbf{u}(\vartheta, \phi) \).

The white mark and the line piercing the sphere surface have the same meaning as in Fig. 4. The two smallest eigenvalues of the covariance matrix \( \text{cov}(\mathbf{q}) \) are almost the same (the relative difference is less than 0.1%) and \( \hat{\sigma}_{\text{max}}/\hat{\sigma}_{\text{min}} \approx 1 \) for modified rotations \( \{ \Delta \mathbf{R} \} \). Fig. 6 shows two angular distributions of vectors \( \mathbf{w}_j(q_j, \tau_j) \) created by applying the measured rotations \( \{ \mathbf{R} \} \) to the vector \( \mathbf{u}_{\text{max}} \) and \( \mathbf{u}_{\text{min}} \) as defined in (1).

Table 1 contains six elements of the covariance matrices \( \text{cov}(\mathbf{q}) \) and \( \text{cov}(\hat{\mathbf{q}}) \) corresponding to the original and modified datasets acquired in noise levels A–C.

Finally, Fig. 7 shows the histograms of the vector components \( q_j = (q^x_j, q^y_j, q^z_j) \) of the unit quaternion \( \mathbf{g}_j = (q^s_j, q^j) \) which parameterize the original \( \{ \Delta \mathbf{R}(\mathbf{g}_i) \} \) and the modified \( \{ \Delta \mathbf{R}(\hat{\mathbf{g}}_j) \} \) random infinitesimal rotations as in (10b).

5. Discussion

The data shown in Fig. 1 have a single peak and Gaussian-like distributions, as expected from repeated measurements under the same experimental conditions. Variances for the noisier data (instrument setting A) are indeed larger than the variances for data acquired for instrument settings B and C. This observation applies to the rotation in the axis-angle representation as well to the (yaw, pitch, roll) parameterization (not shown in this
The histograms of deviation angles $\langle \mu_x, \mu_y, \mu_z \rangle$ shown in Fig. 2 follow indeed the Fisher distribution $G_{\alpha,\beta}$ (3) with Kent correction (4).

Fig. 3 shows the distribution of the rotation angles of random infinitesimal corrections $\Delta R_j$ in axis-angle representation. The distribution of $\rho_j$ follows the general shape of the distributions of deviation angles $\langle \mu_x, \mu_y, \mu_z \rangle$ shown in Fig. 2. As mentioned earlier, this pattern is expected due to (14).

Fig. 4a and c shows the distribution of the axes $\mathbf{a}_j$ of random infinitesimal rotations $\Delta \mathbf{R}_j(\mathbf{a}_j, \rho_j)$. A smallness of the rotational correction $\Delta \mathbf{R}_j$ is gauged only by the rotation
angle $\rho_j$ and, in principle, nothing restricts the axis of infinitesimal rotations to be uniformly distributed on the unit sphere. However, Fig. 4a and c indicate that the axes are highly concentrated along the equator and that they have a bi-modal distribution with two clearly formed peaks. We note that a substantial portion of the sphere surface remains empty (bins labeled with -inf on gray scale). Fig. 4b and d shows the angular uncertainty $\sigma$ has the smallest values concentrated along the equator while large values of $\sigma$ form caps around the poles. This is the result of the fact that if a vector $u$ is roughly parallel to the majority of the axes $a_j$ of random infinitesimal rotations $\Delta R(a_j, \rho_j)$ then such rotations will only cause slight perturbations of the vector $u$. Consequently, the corresponding vectors $w_j = w_j(a_j, \rho_j)$ have a relatively small angular dispersion. The same argument explains why the vectors $u_{max}$ and $u_{min}$ are perpendicular. If the equator is mapped on the surface of a 3D rigid body, it will define the Contour of Smallest Error (CSE). Since the equator is determined by the direction of the eigenvector of the covariance matrix cov($q$), the eigenvector of cov($q$) also defines CSE. Of course, the actual linear uncertainty of a point $r_j$ on the object's surface will be modified by the distance of that point from the center of rotation, $r_j(r_j, \phi_j, \theta_j) = |r_j| |u(\phi_j, \theta_j)$.

Fig. 5a and b were obtained from modified random infinitesimal rotations $\Delta R(\mathbf{a}_j, \rho_j)$ with a uniform distribution of axes $\mathbf{a}_j$. Consequently, there are no empty regions on the unit sphere where the vector $u$ is perpendicular to the majority of axes $q_j$. Moreover, the corresponding angular uncertainty $\sigma$ exhibits very small variations (compare the color scale in Fig. 5b with the scales in Fig. 4b and 4d). This situation may be expected when the off-diagonal elements of the covariance matrix are small. Large absolute values of the cross correlations terms will signal a highly concentrated distribution of axes $a_j$ while small values will correspond to a uniform distribution of the axes $q_j$. Data included in Table 1 strongly support that this is indeed the pattern. We note large differences (two or three orders of magnitude) in the off-diagonal elements derived from the original axes $a_j$ and from the corresponding uniformly distributed $\mathbf{a}_j$. However, the diagonal elements (variances) show much smaller variations most probably due to the fact that the small angle of rotation $\rho_j$ was used both in $\Delta R(a_j, \rho_j)$ and in $\Delta R(a_j, \rho_j)$. We note a significant difference between the dependence of the angular uncertainty $\sigma$ shown in Fig. 5b and the dependences of $\sigma$ shown in Fig. 4b and 4d. While the eigenvector of the covariance matrix cov($q$) still points to the locations $\pm u_{max}$ of $\sigma_{max}$, there are no polar caps nor is the equator defined by the eigenvector, as seen in Fig. 4b and 4d. Since the two smallest eigenvalues of cov($q$) are almost equal, the two corresponding eigenvectors define a plane in Fig. 5b where the new equator can be found. However, this time the angular uncertainty takes the largest values along such a determined equator.

Fig. 6 shows two extreme distributions of the rotated vectors $w_j$ when the vector $u$ in (1) is set to $u_{max}$ or $u_{min}$. Such large variations in the angular uncertainty of points on the surface of a 3D rigid body have important implications for the intelligent use of orientation data acquired.
by perception systems. For example, if there are alternative options for maneuvering an object by a robot arm, it is safer to choose a path for which the critical points that pass close to an existing obstacle are located close to the CSE. Similarly, if there are mechanically equivalent locations for grasping a part, it would be advantageous to select grasp points which lie close to the CSE. Note that the linear uncertainty of the surface points from the center of rotation (which we ignored in this paper) must also be taken into consideration in practical applications. Specifics of the implemented strategy depend on the details of the design of the actual autonomous robot and characteristics of the 3D perception system. Fig. 8 shows one possible example when object orientation is determined by a 3D camera with some angular uncertainty and the color coding corresponds to regions with different propagated uncertainties. The graph was obtained by setting the center of rotation at the center of object’s bounding box. The object was represented by a mesh and for each $k$-th vertex $\mathbf{v}_k$, a corresponding pair of elevation and azimuth angles $(\theta_k, \varphi_k)$ was determined as in Eq. (7). Finally, the same angular uncertainty data $\sigma(\theta_k, \varphi_k)$ shown in Fig. 4b was mapped onto the object’s surface. For grasping the object, it is better to grip the object in the dark blue area since the points in this region have small angular uncertainty. However, if the application requires that the object be gripped at a different location (for example, so that it can be assembled along with other parts) the camera orientation can be changed so that the angular uncertainty $\sigma$ in the region to be grasped is small.

We finish this section with comments on the distributions of the vector components $\mathbf{q}_j = (q^x_j, q^y_j, q^z_j)$ of the unit quaternion $\mathbf{g}_j = (q^s_j, q^i_j)$ which parameterizes the matrix of small random rotations. Graphs (a) and (c) correspond to rotations $\Delta \mathbf{R}(\mathbf{g}_j)$ calculated from the original rotational data with high noise level (A). Graphs (b) and (d) correspond to the rotations $\Delta \mathbf{R}(\mathbf{g}_j)$ with uniformly distributed axes. Histograms of $q^x_j$ are the same as those for $q^y_j$ presented in (a) and (b).

Fig. 7. Histograms of vector components $\mathbf{q}_j = (q^x_j, q^y_j, q^z_j)$ of the unit quaternion $\mathbf{g}_j = (q^s_j, q^i_j)$ which parameterizes the matrix of small random rotations. Graphs (a) and (c) correspond to rotations $\Delta \mathbf{R}(\mathbf{g}_j)$ calculated from the original rotational data with high noise level (A). Graphs (b) and (d) correspond to the rotations $\Delta \mathbf{R}(\mathbf{g}_j)$ with uniformly distributed axes. Histograms of $q^x_j$ are the same as those for $q^y_j$ presented in (a) and (b).

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for which the distribution of the angle reaches its maximum (Fig. 3). The reason why \( q_z \) behaves differently than the two other components \( q_x \) and \( q_y \) is not clear; we suspect it may be related to the particular parameterization of a unit vector in \((7)\).

6. Conclusions

In this paper we discussed how the uncertainty in the orientation of a 3D rigid body propagates to the uncertainty of a point on the object’s surface. This study shows how the distribution of the axes of the random infinitesimal rotations. We examined the covariance matrix of the orientation data and found that the off-diagonal elements of the matrix provide useful clues about the distribution of the axes on a unit sphere. Specifically, large absolute values of the cross-correlation terms imply a highly concentrated distribution of the axes, while small values suggest a uniform distribution of the axes. A concentrated distribution of the axes yields variability in the angular uncertainty of points on the object’s surface, while a uniform distribution yields almost constant angular uncertainty. For a highly concentrated distribution of axes, the eigenvector of the covariance matrix corresponding to the smallest eigenvalue defines the poles on a unit sphere where the angular uncertainty reaches a maximum and the equator where the angular uncertainty reaches a minimum. These characteristics of pose measuring systems, when known in advance and combined with knowledge of the object’s geometry, may help to plan a safer path or a better grip for robotic operations.

References