Bloch oscillations and quench dynamics of interacting bosons in an optical lattice

K. W. Mahmud,1 L. Jiang,1 E. Tiesinga,1 and P. R. Johnson2

1Joint Quantum Institute, National Institute of Standards and Technology and University of Maryland, 100 Bureau Drive, Mail Stop 8423, Gaithersburg, Maryland 20899, USA
2Department of Physics, American University, Washington, DC 20016, USA

(Received 18 October 2013; published 10 February 2014)

PHYSICAL REVIEW A 89, 023606 (2014)

We study the dynamics of interacting superfluid bosons in a one-dimensional vertical optical lattice after a sudden increase of the lattice potential depth. We show that this system can be exploited to investigate the effects of strong interactions on Bloch oscillations. We perform theoretical modeling of this system, identify experimental challenges, and explore a regime of Bloch oscillations characterized by interaction-induced matter-wave collapse and revivals which modify the Bloch oscillations dynamics. In addition, we study three dephasing mechanisms: finite value of tunneling, effective three-body interactions, and a background harmonic potential. We also find that the center-of-mass motion in the presence of finite tunneling goes through collapse and revivals, giving an example of quantum transport where interaction-induced revivals are important. We quantify the effects of residual harmonic trapping on the momentum distribution dynamics and show the occurrence of an interaction-modified temporal Talbot effect. Finally, we analyze the prospects and challenges of exploiting Bloch oscillations of cold atoms in the strongly interacting regime for precision measurement of the gravitational acceleration g.

DOI: 10.1103/PhysRevA.89.023606 PACS number(s): 03.75.Dg, 03.75.Lm, 67.85.—d, 91.10.Pp

I. INTRODUCTION

Ultracold atoms in optical lattices can simulate many of the phenomena associated with electrons in a periodic potential. Compared to real crystals, however, these artificial crystals made from laser light offer versatile control of system parameters such as the lattice depth, geometry, and particle interactions [1]. Furthermore, long coherence times, absence of impurities, and low dissipation make them an ideal system to observe nonequilibrium quantum dynamics [2,3]. One example is the observation of collapse and revival dynamics of bosonic matter wave coherence in a suddenly raised (quenched) optical lattice [4–6]. Another example is the observation of Bloch oscillations, periodic motion in momentum and real space, of ultracold atoms in an accelerating potential [7–9]. These two examples involve two different aspects of nonequilibrium dynamics: the single-particle physics of Bloch oscillations (BOs) and the multi-particle physics of collapse and revival (CR) coherence oscillations which depend on atom-atom interactions [4,10,11].

Bloch oscillations arise when a constant force is applied to particles in a periodic potential [12]. They have been observed in many physical systems, including semiconductor superlattices [13] and ultracold atoms [7,8,14–16]. Bloch oscillations have also been used as a tool to explore band structures and their topological properties [17,18], make precision measurements of gravity [15,16,19], and have been suggested as a probe to identify quantum phases [20,21]. Although the single-particle physics of Bloch oscillations is well understood, there are still open questions regarding the role of particle-particle interactions [22–25].

In this paper, we perform a theoretical study of the dynamics of interacting ultracold bosons in a one-dimensional optical lattice whose axis is vertically aligned with gravity. Transport in two horizontal directions is suppressed. This system, which has been explored in several recent experiments [4–6], is ideally suited for studying the interplay between particle-particle interactions and Bloch oscillation physics. We consider a quench scenario where, starting from an initial superfluid state, the lattice depth is suddenly increased so that tunneling is suppressed, the atom density frozen, and we are in the strong-field regime \( F \gg J \), where \( F \) and \( J \) are the gravitational potential-energy difference and tunneling energy between two neighboring lattice sites, respectively. We show that the gravity-induced Bloch oscillations are strongly modified by interaction-induced matter-wave collapse and revivals.

In a deep lattice with negligible tunneling and higher-band excitation, the dynamics involves on-site phase evolution governed by the competing and independent effects of \( F \) and \( U \), the two-body interaction energy. We study the dynamics in two limits—the strong-\( U \) (\( U > F \)) regime, and the strong-\( F \) (\( F > U \)) regime. Our analysis provides a unified theory for interacting BOs which treats all regimes, and makes predictions that should be within reach of future experiments. Experiments in the strong-\( F \) regime have recently been performed by Meinert et al. [26].

We also investigate three dephasing mechanisms: (i) finite value of tunneling, (ii) effective three-body interactions, and (iii) residual harmonic trapping. In particular, we model in detail the momentum and real-space oscillations of a lattice-trapped superfluid in the presence of both gravity and a background harmonic potential. We find that the dephasing effect due to effective three-body interactions becomes important for the strong-\( U \) regime. When \( J \neq 0 \), we predict that the Bloch oscillations of the center of mass of the atomic cloud should also go through collapse and revivals, demonstrating an interaction-induced effect on quantum transport. We quantify how the presence of a harmonic trap during the dynamics quickly destroys coherence visibility, although we show that there can also be interaction-modified temporal Talbot revivals [27,28].

We are also interested in the prospects for using Bloch oscillations of cold atoms for precision measurement of \( g \). Most experiments have previously focused on the mean-field regime [15,16,19], where up to 20 000 BOs have been
observed, although very recently experiments [26] have operated within the strongly correlated, deep lattice regime. We present estimates for the bounds on the residual harmonic trapping and finite tunneling that should allow observations of up to 50 000 Bloch oscillations.

Most previous studies of BOS of ultracold atoms have used the Gross-Pitaevskii equation to model the mean-field regime when the number of particles per lattice site is on the order of hundreds or thousands [3,14,16]. In contrast, we use the Bose-Hubbard Hamiltonian, and time-evolving block decimation (TEBD) algorithm [29] for our numerical simulations, to model the dynamics when there are a few atoms per lattice site and particle correlations need to be properly accounted for. We also obtain analytical approximations in the limits of coherent states and the Thomas-Fermi regime. Bloch oscillations for interacting bosons in this regime have been studied by Kolovsky and collaborators [22,30–34], and the transport properties of Mott insulators under a constant force [35–37] and superfluids in a Galileo ramp [38] have also been investigated. Our focus here is on regimes where matter-wave collapse and revivals due to interactions is important.

The article is organized as follows. In Sec. II we present our model, define observables, and describe our computational methods. In Sec. III, we briefly consider collapse and revival dynamics in a mean-field theory when the initial state is a coherent state. In Sec. IV, we present our results for the Bloch oscillations of strongly correlated interacting bosons in a vertical lattice. In Sec. V, we investigate dephasing from collapse and revivals due to interactions is important. Finally, we summarize our results in Sec. VII.

II. MODEL AND METHODS

A. System

We consider quasi-one-dimensional bosons in the lowest band of a periodic or lattice potential with period d. We assume that the particles are tightly confined in the two transverse directions such that tunneling and excitations in the transverse directions are negligible. Under these assumptions the system is initially described by the Bose-Hubbard Hamiltonian,

\[
H_I = -J_i \sum_j (a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j) + \frac{U_f}{2} \sum_j n_j(n_j - 1) + \sum_j V_{T,i} j^2 n_j - F_i \sum_j j n_j,
\]

where \(a_j^\dagger, a_j\) are boson creation and annihilation operators at lattice site \(j\), \(n_j = a_j^\dagger a_j\) is the boson number operator, \(J_i\) is the initial tunneling energy (hopping parameter) between nearest neighbors, and \(U_f\) is the initial on-site particle-particle interaction energy. In addition to the lattice potential, we include an external harmonic potential initially parametrized by energy \(V_{T,i}\). The gravitational potential-energy difference between neighboring lattice sites is \(F_i = mgd\), where \(m\) is the atom mass, \(g\) is the acceleration of gravity, and \(d = \lambda/2\) and \(\lambda\) is the wavelength of the laser that creates the periodic potential. For \(^{87}\text{Rb}\) and a laser with \(\lambda = 738\) nm, the gravitational energy is \(F_i/h = 774\) Hz, where \(h\) is Planck’s constant. The tunneling energy \(J_i\) can be tuned by changing the lattice depth [typically \(3E_R\) to \(41E_R\), where \(E_R = \hbar^2 k^2/(2m)\) is the one-photon recoil energy, and \(\hbar = h/(2\pi)\)]. The interaction energy \(U_f\) depends weakly on lattice depth, but can be tuned via a Feshbach resonance [39]. A magnetic-field gradient can be applied to tune the value of \(F_i\), as in [26].

We start with a superfluid ground state in a shallow, vertically aligned optical lattice. A harmonic trap supports the atoms against gravity. The lattice depth is then suddenly increased and simultaneously the harmonic potential is turned off. These steps create a nonequilibrium state of atoms “falling” in the lattice under the influence of gravity and tunable atom-atom interactions.

is \(F_i/h = 774\) Hz, where \(h\) is Planck’s constant. The tunneling energy \(J_i\) can be tuned by changing the lattice depth [typically \(3E_R\) to \(41E_R\), where \(E_R = \hbar^2 k^2/(2m)\) is the one-photon recoil energy, and \(\hbar = h/(2\pi)\)]. The interaction energy \(U_f\) depends weakly on lattice depth, but can be tuned via a Feshbach resonance [39]. A magnetic-field gradient can be applied to tune the value of \(F_i\), as in [26].

We start with a superfluid ground state in a shallow, vertically aligned optical lattice (see Fig. 1). The atoms are initially supported against gravity by a harmonic potential. The linear potential of gravity shifts the minimum of the harmonic well, and there are no Bloch oscillations in the initial ground state. The depth of the optical lattice is then suddenly increased so that tunneling is suppressed. The parameter values before and after the quench are labeled by subscripts \(i\) and \(f\), respectively. In the ideal quench scenario, tunneling is turned off (\(J_i \rightarrow J_f = 0\)), and simultaneously the harmonic trap is switched off (\(V_{T,i} \rightarrow V_{T,f} = 0\), e.g., using the methods in [4]). The lattice ramp-up is assumed fast compared to atom-atom interactions, yet slow enough to prevent excitations to higher bands. The postquench ideal final Hamiltonian is then

\[
H_{\text{ideal}} = \frac{U_f}{2} \sum_j n_j(n_j - 1) - F_f \sum_j j \times n_j,
\]

where \(U_f\) and \(F_f\) denote the interaction and gravitational energy parameters after the quench. These steps create a nonequilibrium state of the atoms “falling” in the lattice. In contrast, in Ref. [26] the system is “quenched” by suddenly changing \(F_i \rightarrow F_f\) by changing an applied magnetic-field gradient. Here we have taken \(F_i = F_f = F\) throughout, since we focus on measuring \(g\).
After a lattice hold time $t_h$, observables are measured either in situ [40] or through time-of-flight imaging [4]. To see the effects of gravity on the atoms, imaging has to be done from the side as opposed to from the top or bottom.

We also model more realistic experimental conditions where there is residual harmonic trapping and finite tunneling. To find the ground states and simulate the time evolution, we use the time-evolving block decimation (TEBD) algorithm [29]. This is a near-exact numerical method where we can control the accuracy of our simulations. The TEBD algorithm is based on a matrix product state ansatz and is equivalent to time-dependent density matrix renormalization group (DMRG) methods.

B. Observables

To analyze the nonequilibrium dynamics, we follow observables giving the center-of-mass position, momentum distribution, zero-momentum occupation, and condensate fraction. The center-of-mass position $x_{\text{cm}}$ (in units of $d$) is determined from density measurements as

$$x_{\text{cm}}(t) = \frac{1}{N} \sum_{j=1}^{L} j \langle n_j(t) \rangle,$$

where $N = \sum_j \langle n_j \rangle$ is total atom number and $L$ is the total number of lattice sites.

The momentum distribution can be measured using time-of-flight expansion and is given by

$$\langle n_k \rangle = \frac{1}{L} \sum_{i,j} e^{i(k(i-j))} g(i,j),$$

where $g(i,j) = \langle a_i^\dagger a_j \rangle$ is the single-particle density matrix (or Green’s function). As a special case, the occupation of the zero-momentum mode is given by $\langle n_{k=0} \rangle = (1/L) \sum_{i,j} g(i,j)$. For Bloch oscillations the momentum peak translates in $k$ space, and we define visibility as the occupation of the peak momentum denoted by $n_{k_{\text{max}}}$.

Finally, we analyze the condensate fraction $f_c$, which is defined as the largest eigenvalue of the single-particle density matrix $g(i,j)$, divided by $N$. This is a measure of the presence of Bose-Einstein condensation in an interacting many-body system [41].

In our treatment of time dependence of observables, we normalize the momentum distribution with its maximum value at initial time and define $\langle \bar{n}_k \rangle = \langle n_k \rangle / \langle n_{k=0} \rangle$ at $t=0$. Similarly, we normalize other observables with the corresponding initial values and define $\langle \bar{n}_{k_{\text{max}}} \rangle$, $\langle \bar{n}_{k=0} \rangle$, and $f_c$, to facilitate comparisons.

III. COHERENT-STATE DYNAMICS

We can obtain analytic expressions for the collapse and revival dynamics if we assume that the initial superfluid is a product of coherent states $|\alpha_j\rangle$ in each site $j$. Such a state could be achieved experimentally if $U_j$ is initially tuned to near zero. If a coherent state is suddenly projected (quenched) into a deep optical lattice, the resulting nonequilibrium state shows collapse and revival in coherence due to interactions, as observed in Ref. [5].

The momentum distribution after the quench is then given by

$$\langle n_k(t) \rangle = \frac{1}{L} \left| \sum_j \langle a_j^\dagger(t) e^{-ikj} \rangle \right|^2 - \frac{1}{L} \left| \sum_j \langle a_j^\dagger(t) \rangle \right|^2 + \bar{n},$$

where $\bar{n} = N/L$ is the average atom occupation per site.

The Hamiltonian governing the postquench dynamics when $J = 0$ is $H_f = H_{\text{ideal}} + V_{T,f} \sum_j \bar{n}_j$, and the annihilation operator in the Heisenberg picture simplifies to

$$a_j(t) = e^{iH_f t/\hbar} a_j e^{-iH_f t/\hbar} = e^{-i(U/j_n + V_{T,f})(j^2 - F_j)\hbar/a_j}.$$

If we assume that the lattice is homogeneous and large, then $\langle a_j(t) \rangle = 0$ and $\langle n_k \rangle = (1/L) \sum_{i,j} \langle a_j^\dagger(t) e^{-i(k(i-j))} \rangle$. From here we can define the quantity

$$v(t) = |\langle a(t) \rangle|^2 = \bar{n} e^{2\cos(U_j t/\hbar) - 1} \times \langle n_k \rangle + \bar{n},$$

where $\bar{n} = |\alpha|^2$ and the CR oscillation period is $T_U = h/U_j$.

When only the gravitational potential is present during the dynamics, the momentum distribution for a homogeneous system is given by

$$\langle n_k(t) \rangle = \left( \frac{1}{L} \sum_{i,j} e^{i(k(i-j))} g(i,j) \right)^2 - v(t) + \bar{n},$$

where $\omega_g = F_0/h$. A similar expression can be derived when the initial density $n_j = |\alpha|^2$ depends on position (e.g., for a Thomas-Fermi initial profile).

When only a harmonic trap is present during the evolution, we can obtain analytic expressions for the dynamics of $\langle n_{k=0} \rangle$ in two different approximations: (i) assuming a homogenous prequench state, that is $n_j = \bar{n}$, and (ii) assuming a Thomas-Fermi initial density profile. For case (i), we obtain

$$\langle n_{k=0}(t) \rangle = 1/L \left| \sum_j \langle a_j^\dagger(t) e^{iV_{T,f} j^2/\hbar} \rangle \right|^2 - v(t) + \bar{n}.$$

For case (ii), we use a Thomas-Fermi profile, $n_j = \bar{n}(1 - \langle V_{T,f} : \mu_j \rangle)^2$, where $\bar{n} = \mu_j / U_j$ and $\mu_j$ is the chemical potential. Taking the continuum limit the sum turns into an integral, and after the change of variables $y = j \sqrt{V_{T,f}/\mu_0}$, we obtain

$$\langle n_{k=0}(t) \rangle = \bar{n} - v(t) + \frac{1}{L} \left| \sum_j \langle a_j^\dagger(t) e^{iV_{T,f} j^2/\hbar} \rangle \right|^2 \times \int_{-1}^{1} y^{D-1} dy \sqrt{1 - y^2} e^{-\bar{n} + i(U_j t/\hbar) V_{T,f} j} .$$

Here $D$ is the dimensionality of the system. We use Eqs. (9) and (10) in Sec. VI to model the early time decay of the zero-momentum occupation, and analyze the effects of a residual harmonic trap on measuring $g$.

IV. QUENCH DYNAMICS IN A VERTICAL LATTICE

In this section, we give a full treatment of the dynamics going beyond coherent-state approximations of Sec. III. We use TEBD numerical simulations to create a realistic initial state for an interacting ($U_j \neq 0$) system that is harmonically
trapped, and study the dynamics. First we treat dynamics under idealized conditions where, after the quench, \( J_f \to 0 \) (no tunneling) and \( V_{T,f} \to 0 \) (no residual harmonic potential). Next we study the case of \( J_f \neq 0 \). We start with analyzing the effects of interactions on Bloch oscillations.

### A. Effects of interactions on Bloch oscillations

The Hamiltonian of interacting atoms in an optical lattice, after the quench, where \( V_{T,f} \to 0 \), is

\[
H_f = -J_f \sum_j (\hat{a}_j^\dagger \hat{a}_{j+1} + \hat{a}_{j+1}^\dagger \hat{a}_j) + H_{\text{ideal}}. \tag{11}
\]

To understand the effects of interactions, let us first analyze the case when \( U_f = 0 \), for which the Hamiltonian is

\[
H_f = -J_f \sum_j (\hat{a}_j^\dagger \hat{a}_{j+1} + \hat{a}_{j+1}^\dagger \hat{a}_j) - F \sum_j j \times n_j. \tag{12}
\]

Without interaction, Eq. (12) describes the single-particle physics of BOs for a tight-binding model \([12,22,31]\). When \( F = 0 \), the eigenstates are delocalized Bloch states, a linear combination of which form Wannier states localized at each lattice site. In the presence of the linear potential \( F \), the energy levels are discrete \( E_j = F j \), and the eigenstates are Wannier-Stark states, that are localized at each site of the tilted lattice. Due to the discrete energy spectrum, the time evolution of an arbitrary initial state is periodic in time with period \( T_B = h/F \). In the Wannier basis \( |l \rangle \), the Wannier-Stark states are given by \(|\psi_l\rangle = \sum_j J_l (J_l/F)|l\rangle\), where \( J_l(x) \) is a Bessel function of the first kind. If a particle is now prepared in a quasimomentum state \( k \) on band \( n \), it will remain on this band and propagate in a way such that \( k(t) = k(0) - F t \hbar \).

In our study here with \( U_f \neq 0 \), we have a further restriction that \( J_f \) is suppressed after the quench, which corresponds to the strong-field regime \((F \gg J_f)\) of BOs. In the limit of \( J_f \to 0 \), discussed in next section, the Hamiltonian is

\[
H = \frac{U_f}{2} \sum_j n_j (n_j - 1) - F \sum_j j \times n_j, \quad J_f \to 0 \text{ implies that the } j\text{th Wannier-Stark states are equal to the Wannier states on the } j\text{th site. This fact makes the effects of a linear potential and interactions separable during the time evolution. The pure-BO Hamiltonian } H_F = -F \sum_j j \times n_j \text{ and the pure-CR Hamiltonian } H_U = U_f \sum_j n_j (n_j - 1), \text{ are both diagonal in a number Fock basis, and hence the BO dynamics is modified by interaction-induced CR oscillations, each occurring on time scales of } T_B \text{ and } T_U. \text{ This in effect gives the zeroth-order result of the effects of interactions on BOs. We identify two regimes: strong } U (U_f > F) \text{ and strong } F (F > U_f), \text{ and the corresponding periods are } T_B \text{ and } T_U, \text{ respectively. However, when } J_f \neq 0, \text{ as we also study in the next section, the Wannier-Stark states are not equal to the Wannier states. Furthermore, the effects of } U_f \text{ are no longer restricted to just on-site evolution, being modified by the tunneling of atoms. This makes the dynamics of } U_f \text{ and } F \text{ nonseparable. Although the detailed dynamics is complex, the periodicity maintains similar features as the zeroth-order results. This clearly identifies the effects of interactions on BOs as having the same physical origin that gives rise to matter-wave collapse and revivals. The equidistant Wannier-Stark ladders turn into a chaotic energy spectrum depending on the ratio of } J, F, \text{ and } U_f, \text{ and an analytical treatment becomes challenging. Our numerical simulations in the next section support all these features.}

### B. Simulations with \( J_f = 0 \)

Figure 2 shows the postquench dynamics in the strong-\( U \) regime. For the initial superfluid, we choose \( U_i/J_i = 3 \) and lattice size \( L = 32 \). The initial atomic cloud, before the quench, is supported against gravity by a harmonic potential. We choose \( V_{T,i} = 0.02U_i \) and \( N = 40 \). Unless otherwise noted, all figures will use these prequench values. To induce Bloch oscillations, the harmonic potential is turned off simultaneously with the lattice ramp. After the quench, we set \( U_f = 5F \), which corresponds to \( F = 774 \text{ Hz and } U_f \approx 4 \text{ kHz for } ^{87}\text{Rb and a laser with } \lambda = 738 \text{ nm. The collapse and revival experiment of Ref. [4] would fall in this strong-}\ U \text{ regime, if gravity in a vertical direction is considered. To our knowledge, no BO experiment has been performed in this regime of strong } F. \text{ Figure 2(a) shows how the quasimomentum distribution manifests two distinct behaviors—a combination of the effects of } F \text{ and } U_f. \text{ The peak or the center of the distribution moves uniformly in momentum space following

**FIG. 2.** (Color online) Bloch oscillation collapse and revival dynamics in the strong-\( U \) regime when \( U_f > F \). Here \( U_f = 5F \). Shown are the dynamics of (a) the quasimomentum distribution and (b) the peak momentum occupation and condensate fraction, as a function of hold time \( t_h \) in the lattice, given in units of \( h/F \). Panel (a) shows that the atomic momentum performs two kinds of evolution: Bloch oscillations (BO) and collapse and revivals (CR) of coherence. The momentum peak travels in quasimomentum space reaching the end of the Brillouin zone at \( k = \pi \), reecting to \( k = -\pi \), and coming back to \( k = 0 \) to perform one Bloch oscillation with period \( h/F \). During this time interval, the momentum peak also collapses and revives with period \( h/U_f \). During collapse, atoms are distributed in quasimomentum over the entire Brillouin zone. The observables in panel (b) also reveal the simultaneous presence of BO and CR. Dynamics of condensate fraction \( \tilde{f}_c \) is not affected by the linear potential.
\(k(t) = k(0) + mgt/h\) resulting from the gravitational acceleration \(g\). When it reaches the Brillouin-zone boundary at \(k = \pi\) it is Bragg scattered to \(k = -\pi\). The motion continues and the peak returns to its original position at \(k = 0\) in one BO period \(T_B = h/\Gamma\). In this case of an infinitely deep lattice (i.e., \(J_f \to 0\)), the atomic spatial density is frozen and the Bloch motion appears only in momentum space. The dynamics is driven by the relative gravitational phase shift \(e^{img\theta/h}\) between each neighboring site. When \(J_f \neq 0\) we get BOs in both coordinate and momentum states as treated in the next section. \(J_f \to 0\) is the limiting case that lets us clearly identify the role of interactions on BOs.

As for the effects of interactions, Fig. 2(a) shows that the momentum peak undergoes collapse and revival oscillations, with revival time \(T_f\). In this example, there are five CR oscillations per BO, as \(U_f/\Gamma = 5\). Here the BO and CR oscillations are decoupled; the analytic expression in Eq. (8) expresses this concisely. If \(U_f = 0\) (no atom-atom interactions, which could be achieved experimentally via a Feshbach resonance), the momentum peak traverses the Brillouin zone with no CR oscillations. The role of interactions in BOs in causing collapse and revivals is the same as the role interactions play in the CR experiments [4,5].

Effects of interactions on CR oscillations can also be seen in a measurement of visibility by monitoring the momentum peak evolution \(\langle \tilde{n}_{k,\text{max}} \rangle\) as depicted in Fig. 2(b). We also plot the condensate fraction \(f_c\) and \(\langle \tilde{n}_{k=0} \rangle\). The time trace of \(\langle \tilde{n}_{k=0} \rangle\) shows signatures of both gravity-induced BOs and interaction-induced CRs. The condensate fraction and visibility are closely related and proportional [42]. The condensate fraction dynamics shows that the interaction-induced quantum depletion is decoupled from the evolution generated by gravity.

Figure 3 shows postquench dynamics in the strong-\(F\) regime, which could be achieved by using a Feshbach resonance to tune atom-atom interactions toward zero. The recent experiment [26] was performed in this regime. The initial superfluid corresponds to \(U_f/J_f = 3\). After the quench we set \(J_f = 0\) and \(F/U_f = 15\). Figure 3(a) shows 15 Bloch oscillations in momentum space for every CR oscillation. Figure 3(b) shows the dynamics of both the peak momentum occupation and zero-momentum occupation, versus hold time. In this regime, collapse and revivals occur over many BO cycles, and the \(\langle \tilde{n}_{k=0} \rangle\) time trace here also reveals both CR and BO dynamics. CRs can be viewed here as interaction-induced dephasing, and subsequent rephasing, of the Bloch oscillations (see also [22]).

**C. Simulations with \(J_f \neq 0\)**

Figure 4 shows the effects of finite tunneling for both strong-\(U\) (\(U_f/F = 5\) in top row) and strong-\(F\) regimes (\(F/U_f = 15\) in bottom row). Panels (a) and (b) show the center-of-mass (COM) oscillations of the atomic density. Red and blue curves show cases with larger \((J_f = 0.1\, U_f)\) and smaller \((J_f = 0.01\, U_f)\) tunneling, respectively. The influence of interactions can be seen in the COM motion. In Fig. 4(a), five small amplitude kinks are visible for every BO cycle, consistent with the parameter choice \(U_f/F = 5\). In Fig. 4(b), the collapse and revival modulation occurs over 15 BO cycles, consistent with \(F/U_f = 15\). We see in these simulations an example of interaction-induced collapse and revivals for real space quantum transport. We note that the spatial amplitude is less than a lattice spacing, for the parameter regimes explored here. The amplitude of the spatial oscillations is proportional to \(J_f\), and depends on the competition of \(U_f\) and \(F\), and can be understood through Stark localization effects.

Figures 4(c) and 4(d) show the BO dynamics of the quasimomentum distribution for \(J_f = 0.1\, U_f\). In the strong-\(U\) regime in panel (c) we observe a rapid decay of the momentum peak caused by atoms tunneling to and interacting with atoms in neighboring sites [43]. Figures 4(e) and 4(f) show the dynamics of the condensate fraction \(f_c\). In panel (e), we see that the larger \(J_f\) value leads to the fastest damping of the condensate. For \(J_f = 0.1\, U_f\) the BO signals in (c) and CR signals in (e) decay significantly within two BO periods. In contrast, the blue curve for small tunneling shows the expected interaction-driven CR oscillations, without significant decay of the revivals. For the strong-\(F\) regime in panels (d) and (f), we see that the decay is much slower over the same time span.

These simulations highlight how a combination of tunneling \((J_f)\) and interactions \((U_f)\) generates true damping. Finite-\(J_f\) allows tunneling to the neighboring sites and \(U_f\) causes interactions with atoms from neighboring sites which changes intersite phase relationships, thus causing the overall decay of oscillations. In single-particle BO physics with \(U_f = 0\) and \(F \gg J\), there is no damping. Similarly, in interacting BO physics with \(J_f = 0\), there is no true damping as the oscillations revive on the two-body time scale \(U_f\) (and three-body time-scale \(U_3\)). It is the combination of \(J_f\) and \(U_f\) in the presence of \(F\) which causes dephasing. The presence of all three energy scales \((J_f,U_f,F)\) causes the
equally spaced Wannier-Stark ladders to split into a chaotic energy spectrum with multitude of avoided crossings. This has been identified in Refs. [22,23,30] as a reason for interaction-induced decoherence. The experiment reported in Ref. [26] explores this phenomenon.

V. DEPHASING MECHANISMS

A. Effective three-body interactions

In a deep lattice there are effective multibody interactions due to collision induced virtual excitations to higher bands [11,44,45]. Quantum phase revival spectroscopy, based on the collapse and revival phenomenon, has been used to detect the presence of effective higher-body interactions [4,6]. Here we examine the influence of effective three-body interactions on the Bloch oscillation CR dynamics. To model this physics, we add to the Hamiltonian in Eq. (2) the effective three-body term

$$H_{3B} = \frac{1}{3!} U_3 \sum_j n_j(n_j - 1)(n_j - 2),$$  \hspace{1cm} (13)$$

where \(U_3\) is the effective three-body interaction energy.

Figure 5 shows the effects of three-body interactions in the strong-\(U\) (left column) and strong-\(F\) (right column) regimes for \(U_f = 5 F\) and \(F = 15 U_f\), respectively. In both cases we set \(U_f/J_f = 3\) so that we consider the same initial state as in the previous section. Postquench, we have \(U_3 = -0.12 U_f\) [45]. In Fig. 5 we plot the dynamics of three observables: the quasimomentum distribution in the first Brillouin zone, the zero-momentum occupation, and the condensate fraction (which is proportional to the visibility).

For the strong-\(U\) case in Figs. 5(a) and 5(c), we see that the revival of the momentum peak after each BO cycle is incomplete, due to the presence of effective three-body interactions. In contrast, in the strong-\(F\) regime shown in the right column of Fig. 5, the three-body interactions lead to only a small modification of the oscillations, over the time interval shown. We quantify this in panels (e) and (f) by comparing \(f_c\) signals over the same time interval, with and
without three-body interactions. The longer-period envelope in panel (e) is due to effective three-body interactions and shows their significant influence in the strong-$U$ regime. In panel (f) the modification in the signal due to effective three-body interactions is minimal on the same time scale. As expected, the influence of effective three-body interactions is far more prominent for the strong-$U$ case. We note that the dephasing due to effective three-body interactions will also show revivals unless that time scale is longer than other dephasing mechanisms [4,44].

B. Residual harmonic confinement

In the system described so far, the atoms are initially supported against gravity by a harmonic potential. To induce Bloch oscillations after the quench, we have assumed that the harmonic potential is turned off simultaneously with the lattice ramp, allowing the atoms to “fall” in the lattice. Alternatively, a sudden change in applied magnetic field (such that $F_f \neq F_i$) can be used to shift the location of potential minimum, as in [26]. The latter can be done with or without a change in the harmonic confinement. In either approach, in practice, there can remain a residual harmonic background $V_{T,f} \neq 0$.

In the CR experiments of Ref. [4], the harmonic potential was minimized using a combination of red- and blue-detuned light. In Ref. [42], a theoretical analysis of CR in a harmonic trap was performed with an inhomogeneous Gutzwiller ansatz formalism, showing rapid dephasing for a strong harmonic background. In the context of observing the Talbot effect with cold atoms [16,27], harmonic confinement is a necessary ingredient. These experiments were analyzed using the Gross-Pitaevskii or discrete nonlinear Schrödinger equation (DNLSE) formalisms appropriate for the mean-field regime with many atoms [16,27]. In this section, we analyze, using the TEBD method, the effects of harmonic trapping both with and without a linear force in the strongly correlated regime.

1. Role of residual confinement without linear potential

First, we consider the effects of only residual harmonic confinement. This scenario can be achieved by suddenly quenching the superfluid without reducing the harmonic background. In Fig. 6, we show the dynamics assuming an initial state with $U_f/J_i = 3$. To differentiate the effects of the harmonic trap from interactions, we show in Fig. 6(a) a density plot of the quasimomentum distribution $\langle n_k \rangle$ setting $U_f = 0$. The figure shows initial dephasing from the harmonic confinement, with rephasing (a full revival) after a period of $T_f = h/V_{T,f}$. There is also rephasing in other quasimomenta at intermediate times, which give an intricate, ordered structure called a quantum carpet [46,47]. Partial (fractional) revivals with two, three, and integer $n$ momentum peaks are seen at $T_f/n$, and there are further revivals symmetrically placed after $T_f/2$. The physics is analogous to the Talbot effect [28] familiar in optics, in which a coherent state experiencing multisite diffraction gives rise to self-similar patterns in the near-field regime. The collapse and revivals in Fig. 6(a) have nothing to do with interactions, as $U_f = 0$, but are due to the quadratic phase relationship $\langle e^{iV_f f t / h} \rangle$ between the neighboring wells. Interestingly, the condensate fraction $f_c$ is not influenced by the interactions. For $U_f = 0$, $f_c$ is constant.

FIG. 6. (Color online) Collapse and revivals dynamics in the presence of a harmonic trap. The quadratic term in the Hamiltonian due to the harmonic potential gives rise to a temporal Talbot effect that is familiar in optics. Panel (a) depicts a density plot of $\langle n_k \rangle$ for a noninteracting ($U_f = 0$) system showing fractional momentum revivals. Panel (b) shows a density plot of $\langle n_k \rangle$ for an interacting system where $U_f/V_{T,f} = 30$. The interactions destroy the Talbot revivals except at times that are integer multiples of $h/U_f$, i.e., at $t = 2, 3, 5, 6, 10$ in units of $h/U_f$, and at symmetric times around the midpoint. Lighter colors denote higher peaks in the momentum distribution while darker shades denote smaller populations. Panel (c) overlays the zero-momentum population for the above two cases. Panel (d) shows the condensate fraction $f_c$ influenced only by the interactions. For $U_f = 0$, $f_c$ is constant.

dynamics in Fig. 6(d) (blue line) shows that there is always a macroscopic occupation of a single quantum state, although the quasimomentum has a fractal nature.

Figure 6(b) shows a density plot of $\langle n_k \rangle$ when the interactions are nonzero and stronger than the harmonic confinement
energy scale \((U_f / V_{T,f} = 30)\). This figure shows the combined effects of both the harmonic potential (with period \(h / V_{T,f}\)) and CR oscillations (with period \(h / U_f\)). For the parameter choice \(U_f / V_{T,f} = 30\) there are 30 CR oscillations per harmonic period, as we can see in the condensate fraction dynamics in panel (d). This plot also shows that the condensate fraction dynamics is not affected by the external harmonic potential. This physics has been explained in Ref. [42]: the single-particle density matrix of an inhomogeneous system is given by a unitary transformation of a homogeneous system, and consequently the eigenvalue time evolution is the same in either a uniform or trapped system. The overall quantum-carpet pattern of quasimomentum dynamics in panel (a) is also seen in the strongly interacting case in panel (b). However, the partial or fractional Talbot revivals that persist for the interacting case must be located at times when the interaction revivals also occur; in this example that happens at factors of 30, i.e., at \(t = 2,3,5,6,10\) in units of \(h / U_f\), and at symmetric times around the midpoint.

Additional insight can be obtained through analysis of the dynamics of the zero-momentum occupation shown in Fig. 6(c). The \(k = 0\) population quickly decays and then revives after period \(h / V_{T,f}\). The red curve \((U_f = 30 V_{T,f})\) and the blue curve \((U_f = 0)\) show that decay of the population, driven by the harmonic background, occurs irrespective of the value of \(U_f\). Analyzing the early time dependence of the population decay yields information on the number of CR or BO cycles that can be readily observed in an experiment. In this example, the decay is so fast that only three CR oscillations can take place before harmonic dephasing dominates.

Finally, we note that there can be two other types of initial spatial shifts that have been neglected in our simulations. The first, trap shift, is due to the displacement of the center of the harmonic trap within a single lattice spacing. The second, cloud shift, is the displacement of the center of the atomic cloud from the center of the trap caused by gravitational sag. Trap shift has been found to influence the dynamics [42] by introducing a linear shift in time in the momentum position in the first Brillouin zone. These effects can be easily scaled away.

### 2. Role of residual confinement including gravitation (or linear) potential

We now analyze the dynamics when the harmonic potential is only partially turned off during the quench, and there is a linear external potential present such that the location of the trap minimum suddenly shifts with the quench. We also include small but finite tunneling \((J_f \neq 0)\). We expect the dynamics to simultaneously manifest gravity-driven BOs, interaction-driven CR oscillations, a harmonic-background-induced Talbot effect, and the effects of tunneling.

Figure 7(a) shows the quasimomentum distribution versus hold time when \(F = 0\) and \(U_f = 2V_{T,f}\). We see the competing effects of the harmonic potential and interactions as described earlier, with the occurrence of fractional revivals. Figures 7(b)–7(d) show the dynamics versus hold time with \(U_f = 2V_{T,f}, F = 60J_f,\) and \(F = 15U_f\) (the strong-\(F\) regime). The revivals in panel (b) are strongly modified by the Bloch oscillations, which cause the momentum peaks to translate uniformly in \(k\) space and reflect at the edge of the Brillouin zone. Figures 7(c) and 7(d) show dynamics of the center-of-mass and condensate fraction, respectively. The real-space oscillations in panel (c) contains signatures of interaction, the linear accelerating potential, the harmonic trap, and finite tunneling. Its Bloch oscillations go through a CR sequence which is suppressed by harmonic trap Talbot revivals at \(\frac{1}{2}h / V_{T,f}\). Panel (d) shows the condensate fraction \(f_c\).
but (again) no dependence on the harmonic confinement. Its dephasing is also unaffected by gravity. In our treatment of BOs in this paper, the example shown here may be most relevant to an actual experimental system since all of these effects will be present in practice, to some degree.

VI. MEASUREMENT OF $g$

Atomic Bloch oscillations have yielded a method for making precision measurements of forces. Gravitational acceleration $g$ has been measured with different degrees of precision with atomic BECs and thermal atoms in a vertical optical lattice [14–16,19,48]. Different aspects of Bloch oscillation physics have been used for attaining high precisions—for example, Ref. [15] used lattice modulation at the fifth harmonic of the Bloch frequency to induce tunneling, and Ref. [16] used a Feshbach resonance to turn off interactions to reduce interaction-induced dephasing due to mean-field nonlinearity. We investigate here the prospects and challenges for the precision measurement of $g$ within a system of strongly interacting bosons in a suddenly quenched vertical optical lattice.

The precision of the measurement of $g$ in a Bloch oscillation experiment depends on the number of BO cycles that can be observed. Another factor is the narrowness of the momentum distribution for the initial and the time-evolved state. The experiment [15,16], performed in the Gross-Pitaevskii (GP) regime, was able to follow the dynamics for 10–20 s and observe approximately 20 000 BO cycles. On the other hand, experiments on CR [4] followed the dynamics for 20 ms, long enough time for 20–30 BO cycles. Reasons for the fast decay of signals in the CR experiments could include pumping energy into the system due to the quench, presence of a residual harmonic trap, presence of finite tunneling, three-body loss and other interaction related losses, and the effect of changing of the Wannier function [49]. Here we analyze the effects of a harmonic trap and finite tunneling, and calculate bounds on their values for observing up to 50 000 BO cycles.

A. Bounds on residual harmonic trap

We have shown in Sec. V that the presence of a residual harmonic trap during the dynamics causes rapid decay in $\langle n_k \rangle$. In separate experiments involving BOs [27] and CRs [4], harmonic trap effects were minimized; however, a number was not given on how small the value is. Even a minute trap strength can have a significant effect over many oscillations. Figure 8(a) shows a TEBD simulation of the early time decay of $\langle \hat{n}_k \rangle$ with $V_{T,f}/F = 0.002$, for $F = 2$, $U_f = 10$, and $L = 32$, when the initial state is homogeneous $V_{T,f} = 0$, and final tunneling is suppressed $J_f = 0$. We see that both the Bloch and CR oscillations occur on time scales $h/F$ and $h/U_f$, respectively, and they decay due to the residual harmonic trap, following a common envelope function. The envelope function can be understood from the approximate continuum form of Eq. (9) assuming an initial superfluid that is homogeneous and coherent, and evolving in a harmonic trap of strength $V_{T,f}$. The effects of $V_{T,f}$, $U_f$, and $F$ are separable and the initial decay of the envelope function is given by the analytical expression

$$\langle n_{k=0}^{\text{env}}(t_h) \rangle \propto \frac{\text{erf}(ie^{i\pi/4}\alpha)}{4\alpha^2}, \tag{14}$$

where erf is the error function, $\alpha = L\sqrt{V_{T,f}t_h/(2h)}$, $L$ is the number of lattice sites, and $t_h$ is the hold time. Figure 8(b) shows $\langle n_{k=0}^{\text{env}} \rangle$ as a function of the dimensionless variable $\alpha$. The expression is universal for any harmonic trap and lattice size; decay for different trap strengths can be scaled to fall on this same curve. A comparison of analytical result with TEBD simulations, which include correlations due to tunneling in the initial superfluid, shows a good match for the initial decay. Panel (c) shows a log-log plot of the number of BO cycles when $\langle \hat{n}_{k=0} \rangle$ drops to $1/e$ of its initial value, as a function of residual harmonic trap strengths for lattice sizes $L = 64$ (red curve) and 100 (blue curve). The dots and arrows on the $V_{T,f}/F$ axis denote the corresponding bound for 50 000 BO.

Fig. 8. (Color online) Bounds on the number of Bloch oscillations due to a residual harmonic trap. Panel (a) depicts a TEBD simulation, with $L = 32$, of the effects of a trap for $V_{T,f}/F = 0.002$, $F = 2$, and $U_f = 10$. We see that both the Bloch (dotted line) and CR (full line) oscillations with period $h/F$ and $h/U_f$ are modified by a decay envelope characteristic of the trap strength. Here we show only the short-time dynamics where the trap-induced decay of visibility takes place. In Panel (b), we show the analytical envelope of $\langle \hat{n}_{k=0} \rangle$, Eq. (14), assuming that the initial superfluid state is a coherent state and homogeneous, as a function of scaled time $\alpha = L\sqrt{V_{T,f}t_h/(2h)}$. We compare this to TEBD results with $L = 32$ showing a good match for the initial decay. Panel (c) shows a log-log plot of the number of BO cycles when $\langle n_{k=0} \rangle$ drops to $1/e$ of its initial value, as a function of residual harmonic trap strengths for lattice sizes $L = 64$ (red curve) and 100 (blue curve). The dots and arrows on the $V_{T,f}/F$ axis denote the corresponding bound for 50 000 BO.
the envelope of $\langle n_{k=0}(t) \rangle$ decays to $1/e$ of its initial value. This gives a relationship between the background trap strength $(V_{T,f}/F)$ and $N_B$,

$$N_B \approx \frac{1}{2\pi} \frac{e^2 \sqrt{\pi}}{L^2} \frac{F}{V_{T,f}}.$$  \hspace{1cm} (15)

In Fig. 8(c) we plot the value of $V_{T,f}/F$ needed to observe $N_B$ Bloch oscillations for lattice sizes $L = 64$ and $100$; note that it is a log-log plot. The filled circles represent the point for 50 000 BOs and the arrows below indicate the trap strengths required. For the experimentally relevant lattice sizes between $L = 50$ and 100, the trap strength needs to be extremely small, e.g., $V_{T,f}/F \approx 10^{-8}$, to observe BO cycles beyond the current maximum value of 20 000 [27]. Larger lattice sizes make the constraint more severe.

If the initial frequench state is trapped, it causes density inhomoegeneity and $N_B$ increases up to 20%. Then $N_B$ depends on a combination of initial trap parameters, total atom number, and the density profile, and specific cases must be analyzed numerically. We note that the momentum width for an initially trapped system is bigger, and it spreads more quickly during the dynamics, eventually making the number of observable BOs smaller. The overall effect of an initial trap is not significant compared to that of the residual trap, and hence our analysis here using an initially homogeneous density profile gives a good approximation for the bounds on $V_{T,f}$.

B. Bounds on finite- $J$ effects

In an ideal BO collapse and revival scenario $J_f = 0$ and the momentum peak revives completely. For $J_f \neq 0$, Fig. 4 showed that CR oscillation amplitudes slowly decay, and similarly the BO signal dephases due to the competition among $J_f$, $U_f$, and $F$. Here we analyze the bounds on the number of Bloch oscillations for a finite nonzero value of $J_f/U_f$ and $J_f/F$. For this, we assume that $J_f$ is small, such that $F \gg J_f$ and $U_f \gg J_f$.

In Fig. 9 we analyze the decay of CR oscillations of $\langle \tilde{n}_{k=0} \rangle$ for different values of $J_f/U_f$ and $F = 0$, $V_{T,f} = 0$, using TEBD numerical simulations. The initial superfluid state is a homogeneous coherent state with $U_i/J_i = 0$, $\tilde{n} = 1.5$ and $L = 32$. Figure 9(a) shows the dynamics in units of interaction time scale $h/U_f$ where the expected higher rate of decay for larger tunneling values is evident. If the same data are plotted in units of $h/J_f$, as shown in panel (b), a common envelope function is seen to characterize the decay of $\langle \tilde{n}_{k=0} \rangle$. This implies that for a specific value of tunneling $J_f/U_f$, the signal decay after one oscillation is equal to that of the $M$th oscillation for the smaller value $h/J_f$. The reference $J_f/U_f$ needs to be small for this relationship to hold. For values considered in Fig. 9(b) this holds true. The damping of the first oscillation revival analyzed in Refs. [50,51] is consistent with our findings. We show here that this analysis can be extended to the $M$th oscillation, and propose a method to estimate the number of observable oscillations for a smaller tunneling rate by calculating the oscillation decay for a larger one.

We can also make a connection with the number of observable BOs, again defined by the time at which the envelope of $\langle \tilde{n}_{k=0} \rangle$ reaches $1/e$ of its initial value. In the presence of finite $J$, the CR oscillations due to $U_f$ and BOs due to $F$ are coupled as discussed in Sec. V B. In the regime of interest, when $F \gg J_f$ and $U_f \gg J_f$, the effects of $U_f$ and $F$ on the oscillations are approximately separable. Figure 9(c) depicts the Bloch oscillations $N_B$ that can be observed for different values of $J_f/F$. To observe 50 000 BO cycles, the $J_f/F$ value needs to be $0.8 \times 10^{-6}$. The value of $J_f$ depends on optical lattice depth $V$ in the following way:

$$\frac{J_f}{U_f} = \frac{2\pi a_s \hbar}{V},$$

where $V$ is given in units of recoil energy $E_r$, $a_s$ is the $s$-wave scattering length, and $d$ is the lattice spacing. For specific atomic species and lattice setup, a value of lattice depth can be determined such that these criteria are satisfied.
The damping of BOs due to finite $J$ depends on several things: the initial average occupation $J_f/U_f$, initial trap strength $V_{J_f}$, and the force $F$, in addition to its dependence on $J_f/U_f$. We find here that knowing all the other parameters, the bounds on $J_f/F$ and $J_f/U_f$ to observe $N_B$ oscillations has a linear dependence. We have not discussed even longer term behavior of the decay as the question of thermalization and equilibration can become important [52,53]. The value of $J_f/U_f$ should be small in precision measurement experiments such that a large number of Bloch oscillations can be observed before equilibration takes place.

VII. CONCLUSION AND SUMMARY

In this paper, we have shown that the effect of multiparticle interactions on Bloch oscillations physics is described by the physics of matter-wave revivals—full revivals for a decoupled lattice and partial revivals for a coupled lattice, all occurring on the interaction time scale. We performed a theoretical analysis of interacting ultracold bosons in a suddenly ramped one-dimensional optical lattice that is vertically aligned. This system can be systematically tuned and exploited to study the effects of interactions on BOs. We used the Bose-Hubbard Hamiltonian to model the dynamics in the strongly interacting regime, and studied the dynamics in two limits—the strong-$U$ ($U > F$) regime, and the strong-$F$ ($F > U$) regime, where $U$ and $F$ are respectively atom-atom interaction and linear potential strengths, after the quench. We have used the time-evolving block decimation (TEBD) algorithm for our numerical simulations.

We analyzed three dephasing mechanisms for the oscillations—finite value of tunneling $J$, effective three-body interactions, and residual harmonic trapping. We find that the dephasing effect due to effective three-body interactions becomes important for the strong-$U$ regime. When $J \neq 0$, we predict that Bloch oscillations of the center of mass of the atomic cloud should also go through collapse and revival modulations, demonstrating an example of quantum transport where real-space revivals occur. We also show that the presence of a harmonic trap during the dynamics quickly destroys coherence visibility of the atoms and gives rise to a temporal Talbot effect [27,28], which survives in the strongly interacting few-atom regime. We further model in detail the momentum and real-space oscillations of a lattice-trapped superfluid in the presence of gravity, a residual harmonic potential, and finite tunneling.

In addition to studying the interplay between interactions and Bloch oscillations physics, we examine the prospects of measuring gravitational acceleration $g$ with high precision using a system of strongly correlated ultracold atoms in a deep lattice. We present numerical and analytical results for error bounds on the residual harmonic trap and finite tunneling to go beyond the current maximum observation of 20 000 Bloch oscillations. The analysis and characterization of a realistic experimental system is a necessary step towards the goal of surpassing the current precision limit of $g$.

The ideas investigated here are extremely relevant in the light of current experimental efforts [26]. Further insights can be gained by studying more comprehensively the competition of $F$, $U$, and $J$. Realistic conditions such as finite temperature and higher-band effects may also be relevant for cold atom experiments. The effects of interactions on Bloch oscillations in cold atoms and other systems deserve additional analysis for the exploration of fundamental physics as well as measurement applications.

ACKNOWLEDGMENTS

We acknowledge support from the US Army Research Office under Contract No. 60661PH and the National Science Foundation Physics Frontier Center located at the Joint Quantum Institute.