Spin dephasing as a probe of mode temperature, motional state distributions, and heating rates in a two-dimensional ion crystal

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We employ spin-dependent optical dipole forces to characterize the transverse center-of-mass (COM) motional mode of a two-dimensional Wigner crystal of hundreds of $^9$Be$^+$. By comparing the measured spin dephasing produced by the spin-dependent force with the predictions of a semiclassical dephasing model, we obtain absolute mode temperatures in excellent agreement with both the Doppler laser cooling limit and measurements obtained from a previously published technique [B. C. Sawyer et al., Phys. Rev. Lett. 108, 213003 (2012)]. Furthermore, the structure of the dephasing histograms allows for discrimination between initial thermal and coherent states of motion. We also apply the techniques discussed here to measure the ambient-heating rate of the COM mode of a 2D Coulomb crystal in a Penning trap. This measurement places an upper limit on the anomalous single-ion heating rate due to electric field noise from the trap electrode surfaces of $\bar{\alpha}_s \sim 5$ s$^{-1}$ for our trap at a frequency of 795 kHz, where $\bar{n}$ is the mean occupation of quantized COM motion in the axial harmonic well.

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1. INTRODUCTION

Laser-cooled ions stored in radio frequency (RF) or Penning traps readily form crystalline arrays. Sensitive measurements of the motion of ions in these arrays are important for a variety of studies in atomic physics, quantum information science, and plasma physics. In atomic physics the small residual motion of trapped ions produces systematic errors that can limit the performance of atomic clocks and precision measurements [1,2]. In quantum information science, where trapped-ion crystals provide a promising platform for quantum computation and simulation [3–7], the residual motion of the trapped ions produces infidelities that require careful evaluation. In plasma physics, trapped-ion crystals provide a convenient laboratory platform for studies of strongly coupled plasmas, which model dense astrophysical matter [8–11]. Careful measurements of ion motion are used to determine the ion energy and the plasma coupling. Energy transport studies require detailed measurements of the ion motion, typically as a function of time and resolved spatially or between different modes [12].

Here we discuss a new technique for measuring the temperature and, more generally, the energy state distribution of a trapped-ion crystal. The technique is mode specific in that it can resolve the energy distribution of different modes of the crystal. We demonstrate the technique by presenting measurements of the energy distribution of the axial center-of-mass (COM) mode of a single-plane array of several hundred Be$^+$ ions stored in a Penning trap. The technique requires isolating and controlling a two-level system—an effective spin-1/2 in each ion—and employs a weak, global spin-dependent force that couples the spin and motional degrees of freedom of each ion. This general technique should be applicable to other systems such as neutral atoms in optical lattices [13] or optomechanical systems [14] where spin degrees of freedom can be controlled and coupled to motional degrees of freedom.

Spin-dependent forces are a key tool in trapped-ion quantum simulation and quantum computing work. Application of a spin-dependent force to a superposition of different spin states can generate entanglement between the spins, while the concomitant coupling of the spin and motional degrees of freedom typically produces infidelities that must be mitigated [15–21]. Here we focus on the coupling and potential entanglement between the spin and motional degrees of freedom produced by a spin-dependent force, and work in a regime where the induced spin-spin entanglement is negligible. However, the discussion and measurements presented here provide insight into the size and nature of trapped-ion quantum gate errors produced by coupling of the spins to thermal fluctuations of the motional modes [22]. Spin-echo as well as other dynamical decoupling techniques can remove the coupling of the spin and motional degrees of freedom [16,23,24], but their efficacy depends on the size of the error and the coherence of the motional state throughout an experiment, which can be evaluated with the techniques discussed here.

This study extends the results of Ref. [15], where we measured the decrease in the composite Bloch vector length produced by the application of a homogeneous spin-dependent optical dipole force. We showed that this decrease (or decoherence) of the Bloch vector depended on the average energy or temperature of the initial motional state. Here we show that the dephasing responsible for this decoherence may be directly measured, revealing more detailed information about the motional state. In addition to the average energy or temperature of a mode, information on the energy distribution can also be obtained. Spin dephasing produced through the application of a spin-dependent force provides an alternative to the well known Raman sideband technique for determining the energy distribution of motional states of trapped-ion crystals [25]. The spin-dephasing technique is particularly well suited for many-ion crystals, and for some setups—in particular for higher frequency two-level systems such as the 124 GHz spin-flip transition discussed here (see Sec. II)—can be simpler to implement.

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To illustrate the basic idea of spin dephasing produced through the application of a spin-dependent force, we consider the simple case of a single trapped ion whose motional degree of freedom along the \( z \) axis (trap frequency \( \omega_z \)) is coupled to two internal spin states through a sinusoidally time-varying spin-dependent force. The interaction Hamiltonian for this system is

\[
\hat{H} = F_0 \cos (\mu t) \hat{z} \delta z, \tag{1}
\]

where \( \hat{z} \) is the position operator of the ion in the \( z \) direction, \( \delta z \) is the Pauli spin matrix associated with the two internal energy levels, and \( \mu \) is the frequency of the applied spin-dependent force. We assume the ion spin state is initialized in an equal superposition \( \frac{\ket{\uparrow} + \ket{\downarrow}}{\sqrt{2}} \) of the \( \ket{\uparrow}, \ket{\downarrow} \) internal levels. This spin state can be represented as pointing along the \( x \) axis in the rotating frame of the Bloch sphere. Suppose the ion temperature is large compared to \( h \omega_z / k_B \), where \( h \) and \( k_B \) are the Planck and Boltzmann constants, and we may treat the ion motion as classical. The initial motional state of the ion can then be written as \( z(t) = Z_A \cos (\omega_z t + \phi) \), where \( Z_A \) and \( \phi \) fluctuate from one shot (or realization) of the experiment to the next, consistent with a thermal distribution. Application of Eq. (1) produces an additional spin-dependent motion, but we assume this driven spin-dependent motion is small compared with the initial thermal fluctuation \( (Z_A) \) (valid for our work with hundreds of trapped \( ^9 \text{Be}^+ \) ions), and we can approximate the Hamiltonian \( \hat{H} \) as

\[
\hat{H} \approx F_0 Z_A \cos (\omega_z t + \phi) \hat{z} = \frac{F_0 Z_A}{2} \left[ \cos (\mu - \omega_z) t - \phi) + \cos (\mu + \omega_z) t + \phi) \right] \hat{z}. \tag{2}
\]

For \( \mu = \omega_z \) this Hamiltonian is simply a constant shift \( F_0 Z_A \cos (\phi) / h \) in the frequency difference between the \( \ket{\uparrow}, \ket{\downarrow} \) levels, plus a rapidly varying term that averages to zero for time intervals long compared to \( \pi / \omega_z \). If the spin-dependent force is applied for a time interval \( \tau \), then the Bloch vector undergoes a precession by an angle \( \Phi \approx \frac{F_0 Z_A \cos (\phi)}{\hbar} \tau \). Fluctuations in \( Z_A \) and \( \phi \) from one shot (or realization) of the experiment to the next produce spin dephasing when averaged over many experimental realizations. By measuring this dephasing directly we show that it is possible to acquire information on the initial motional state (for example, the energy distribution) of the trapped-ion harmonic oscillator. The sensitivity to motion of this technique is very high. For the modest parameters used in the measurements of Sec. III and IV, \( F_0 = 10^{-23} \) N and \( \tau = 1 \) ms give \( \Phi = 30^\circ \) for \( Z_A \approx 6 \) nm. With \( N \gtrsim 100 \) trapped ions, a 30° precession is much larger than the quantum projection noise [26], and can be measured with good signal-to-noise in one experimental shot.

The rest of the manuscript is structured as follows. In Sec. II we describe the Penning trap setup where we implement the spin dephasing technique to characterize the energy distribution of the axial COM mode of a 2D trapped-ion crystal of hundreds of \( ^9 \text{Be}^+ \) ions. In Sec. III we discuss a more detailed dephasing model assuming a thermal distribution of coherent states. In Sec. IV we discuss dephasing measurements of the COM mode energy distribution for both thermal states and coherently excited thermal states. We also present measurements of the heating rate of the axial COM mode. In Sec. V we summarize and conclude.

## II. EXPERIMENTAL SETUP

As described in previous publications, we employ a Penning trap to confine crystals of hundreds of \( ^9 \text{Be}^+ \) ions [7,15]. Depicted in Fig. 1(a), the trap consists of a static electric \( (E) \) quadrupole produced from a stack of cylindrical electrodes (inner radius of 2.0 cm) placed within the room-temperature bore of a \( \approx 4.46 \) T superconducting magnet. The orientation of this uniform magnetic \( (B) \) field defines the \( z \) axis in our system. Harmonic axial \( (z) \) axis ion confinement with a frequency of \( \omega_z = 2 \pi \times 795 \) kHz is obtained by applying \( -1 \) kV to the central ring electrodes relative to grounded upper and lower end cap electrodes. The cylindrical axis of the trap electrodes is aligned with the uniform magnetic field. Radial ion confinement results from \( \vec{E} \times \vec{B} \) induced rotation through the magnetic field. We apply a weak quadrupolar “rotating wall” potential to precisely control the rotation frequency \( (\omega_r) \) and hence, radial confining force, of the ion cloud [27]. Neglecting the weak azimuthal dependence of the rotating wall potential, the following trap potential describes ion confinement in the Penning trap as seen in a frame rotating at \( \omega_r [28] \):

\[
q \Phi_{\text{trap}}(r,z) = \frac{1}{2} M \omega_z^2 (z^2 + \beta_r r^2), \tag{3}
\]

\[
\beta_r = \frac{\omega_r \omega_z}{\omega_z^2} - \frac{1}{2}, \tag{4}
\]

![FIG. 1. (Color online) (a) Simplified illustration of the Penning trap electrode structure in cross section (not to scale). Static voltages are applied to the electrodes in the foreground (orange), while three of the six rotating wall electrodes (red) are shown in the background. A simulated single-plane ion configuration is shown at the trap center and magnified by \( \approx 100 \) for visibility. The 313 nm intersecting optical dipole force beams are also illustrated. (b) Low-lying electronic levels of \( ^9 \text{Be}^+ \) in the 4.46 T \( B \) field of the Penning trap. Projection of total electronic angular momentum \( (m_J) \) is given to the right of each level. Nuclear spin projections have been excluded for clarity. Relevant transitions near 313 nm are labeled.](image-url)
where \( M \) (\( q \)) is the mass (charge) of a single \(^9\text{Be}^+\), \( \Omega_c = 2\pi \times 7.597 \text{ MHz} \) is the cyclotron frequency, and \( z \) (\( r \)) is the axial (radial) distance from the trap center. For rotation frequencies where the radial confinement is weak relative to transverse confinement (\( \beta \ll 1 \)), we obtain a single ion plane. The rotation frequency at which the ion configuration transitions from two planes to one plane depends sensitively on the ion number [29]. For most experiments, we operate with 100 to 300 ions, which necessitates \( \omega_c \lesssim 2\pi \times 48 \text{ kHz} \) for single-plane conditions with \( \omega_c = 2\pi \times 795 \text{ kHz} \).

We use Doppler laser cooling along both the axial (\( z \) axis, out-of-plane) and radial (in-plane) trap dimensions to produce the Coulomb crystal. As shown in Fig. 1(b), the hyperfine structure of \(^9\text{Be}^+\) exhibits a strong Zeeman shift in the large uniform \( B \) field of the Penning trap. We laser cool along the \( ~\sim 313 \text{ nm} \) \( |J = 1/2, m_J = +1/2 \rangle \rightarrow |3/2, +3/2 \rangle \) cycling transition between the \( ^3S_{1/2} \) and \( ^2P_{3/2} \) manifolds, where \( J \) and \( m_J \) are the total electronic angular momentum and its projection along the \( B \)-field axis, respectively. The linewidth of this cooling transition is \( \Gamma \sim 2\pi \times 17.97 \text{ MHz} \) [30], yielding a Doppler cooling limit of \( \hbar \Gamma / 2k_B \sim 0.43 \text{ mK} \).

For all experiments described here, the \(^9\text{Be}^+\) are optically pumped to the \( |I = 3/2, m_J = +3/2 \rangle \) nuclear spin state [31].

The two qubit states for our experiments are the \( |\uparrow\rangle \equiv |1/2, +1/2 \rangle \) and \( |\downarrow\rangle \equiv |1/2, -1/2 \rangle \) valence electron spin projections of the \( ^3S_{1/2} \) electronic ground state. The cooling and repump transitions illustrated in Fig. 1(b) allow for efficient preparation of all \( N \) trapped ions to the state \( |\uparrow\rangle_N \equiv |\uparrow \uparrow \ldots \uparrow \rangle \). The splitting between qubit levels is \( \sim 124 \text{ GHz} \), and we perform global qubit rotations via direct application of resonant millimeter-wave radiation to the ions. We typically achieve \( \pi \)-pulse times \( (t_\pi) \) of \( \sim 70 \mu s \). As discussed below, for the experiments described here we perform global readouts of the qubit state through state-dependent resonance fluorescence on the Doppler cooling transition.

Figure 2(a) illustrates a typical pulse sequence for qubit manipulation. We first prepare \( \left|\uparrow\right\rangle_N \) using the Doppler cooling and repump lasers. The spin echo sequence shown includes both \( \pi /2 \) and \( \pi \) pulses about the given Bloch sphere axes. The phase of the first pulse (\( \Delta \phi \)) is defined relative to that of the first \( \pi /2 \) pulse, which we define to be a rotation about the \( y \) axis, and is varied depending on the intended final spin state. We use a spin echo sequence with free evolution periods of \( t \sim 0.1 \) to 1 ms to mitigate the deleterious effects of radial \( B \)-field inhomogeneity over the \( \sim 400 \mu \text{m} \) ion plane diameter, and to cancel \( B \)-field fluctuations at frequencies below \( \tau^{-1} \) [32,33]. After the pulse sequence, we measure the population of spins in state \( |\uparrow\rangle \) (\( P_\uparrow \)) by switching on the Doppler cooling beams and counting scattered photons collected by an ultraviolet-sensitive photomultiplier. An \( f /3 \) objective imaging the side of the ion plane collects the scattered cooling photons, and a typical photon count rate per ion is \( 10^3 \text{ s}^{-1} \).

The histograms of Fig. 2(b) give experimental results for three different values of \( \Delta \phi = \pi /2 \) as measured in a system of \( N = 174(10) \) spins. Each color-coded histogram is the result of 1000 pulse sequences [see Fig. 2(a)] and subsequent qubit state readouts. Bin widths for the three histograms are adjusted for clear presentation on a single vertical scale, and the horizontal axis is scaled to the photon counts collected for the state \( |\uparrow\rangle_N \).

The standard deviation of the \( \Delta \phi = \pi \) histogram suggests technical noise that is comparable to shot noise with 3510 photons collected. The increase in the standard deviation of the \( \Delta \phi = \pi /2 \) data is due to quantum spin projection noise (\( \propto N^{-1/2} \)) [26].

We generate a spin-dependent optical dipole force (ODF) by interfering two \(~\sim 313 \text{ nm} \) laser beams at the ion plane to create a one-dimensional optical lattice [see Fig. 1(a)] [7,15]. The resulting lattice wavelength (\( \lambda_l \)) is determined by the crossing angle of the ODF beams as \( \lambda_l = \lambda_{\text{ODF}}[2 \sin(\Delta \phi / 2)]^{-1} \), where \( \theta_B \) is the full beam crossing angle and \( \lambda_{\text{ODF}} \) is the ODF laser wavelength. For this work, \( \theta_B = 4.2(2) \)°, which results in \( \lambda_l \sim 3.7 \mu \text{m} \). The ODF laser frequency is detuned by \( \sim 20 \text{ GHz} \) from the nearest resonances for the \( |\uparrow\rangle \) and \( |\downarrow\rangle \) states [see Fig. 1(b)] and the linear polarization of each beam is chosen so as to produce a polarization gradient at the ion plane that imparts equal-magnitude, opposite-sign forces to the two qubit states with a magnitude of \( \sim 10^{-23} \text{ N per spin at } 1 \text{ W cm}^{-2} \) per beam. The optical lattice wave vector (\( \Delta k \)) is aligned to within 0.05° of the \( z \) axis of the trap to preferentially excite motion transverse to the crystal plane and to maximize homogeneity of the ODF drive phase across the sample. The two ODF laser beams are produced from a single beam using a 50:50 beam splitter, and their relative frequency is adjusted from zero to \( \sim 10 \text{ MHz} \) using acousto-optic modulators, enabling production of a standing- or running-wave spin-dependent optical lattice.
III. THERMAL DEPHASING MODEL

In Ref. [15] we excited arbitrary drumhead modes of a 2D trapped-ion crystal through application of a homogeneous spin-dependent force. The spin-dependent force coupled the $^9$Be$^+$ ground state valence electron spin and transverse motional degrees of freedom. We measured the decrease in the composite Bloch vector of the spins due to this coupling, and showed that this decrease (or decoherence) depended on the average energy or temperature of the motional state. We sketch the calculation of Ref. [15] in Appendix A. In the Fock state basis thermal motional states are described by a diagonal density matrix. The calculation proceeds by assuming an initial Fock state $|n\rangle$ for a mode. Application of a spin-dependent force produces spin-dependent displacements of the Fock state and decoherence of the spins is naturally described in terms of spin-motion entanglement and the increasing displacement sensitivity of Fock states with $n$. Here we use a model motivated by the dephasing picture of the Introduction that does not require quantum entanglement of the spin and motional degrees of freedom for thermal excitations large compared with the ground state size.

For simplicity, we describe the dephasing model for the axial COM mode, although a generalization to other drumhead modes is straightforward. Center-of-mass motional modes play an important role in quantum information experiments with trapped ions. If all trapped ions possess the same charge-to-mass ratio, the COM mode frequency is independent of ion number and, in the case of transverse modes, constitutes the highest-frequency and longest-wavelength oscillation. For the experimental parameters presented in this article, the COM mode is $> 10$ kHz from the nearest transverse modes and is well resolved [21]. In traps whose electrode dimensions are much larger than those of the ion crystal (i.e., Penning traps), the COM mode is the transverse mode most susceptible to noise from fluctuating potentials on trap electrodes.

The interaction Hamiltonian for the spins and the COM mode is:

$$\hat{H}_{\text{ODF}} = F_0 \cos(\mu t + \varphi) \frac{z_0}{\sqrt{N}} (\hat{a} e^{-i\omega_0 t} + \hat{a}^\dagger e^{i\omega_0 t}) \sum_{i=1}^{N} \hat{\delta}_i^z,$$

(5)

where the sum is over the $N$ spins, $z_0 = \sqrt{\hbar/(2Ma_0)}$ is the ground state wave function size of a single trapped ion, $\hat{a}$ ($\hat{a}^\dagger$) are the lowering (raising) operators for the COM mode, and $\varphi$ is the ODF phase. In general the time evolution operator for the above Hamiltonian can be written as the product of a spin-dependent displacement operator, $\exp(\alpha \hat{a}^\dagger + \alpha^* \hat{a})$, and an evolution operator for a general Ising interaction that involves only pairwise spin interactions [15,17]. For resonant drives ($\mu \approx \omega_z$), the effect of the spin-dependent displacement typically dominates, and we neglect the induced Ising interaction throughout this manuscript. In this case the evolution operator for $\hat{H}_{\text{ODF}}$ separates into a product of $N$ individual spin-dependent displacement operators,

$$\hat{D}_{\text{SD}}(\alpha) = \prod_{i=1}^{N} \exp[(\alpha \hat{a}^\dagger - \alpha^* \hat{a}) |\delta_i^z\rangle].$$

(6)

The displacement amplitude $\alpha$ for resonant ($\mu = \omega_z$) spin-dependent excitation of the COM mode for a time $\tau$ and phase $\varphi$ is

$$\alpha(\tau, \varphi) = -i \frac{F_0 z_0 \tau}{2\hbar \sqrt{N}} e^{i\varphi}. \tag{7}$$

For pulse sequences involving separated periods of ODF excitation (e.g., spin echo), the final motional displacement is simply a sum of individual displacements of the form of Eq. (7) with appropriate phases ($\varphi$) and times ($\tau$) for each of the ODF excitations within the sequence. We define spin-independent displacements, $\hat{D}(\alpha_0)$, in the usual way:

$$\hat{D}(\alpha_0) = \exp(\alpha_0 \hat{a}^\dagger - \alpha_0^* \hat{a}). \tag{8}$$

In contrast to Ref. [15], we consider the initial state of the COM mode for each experiment to be a coherent state $|\alpha_0\rangle$. We denote the expectation value of a quantum operator $\hat{O}$ at the end of an experiment as $\langle \hat{O} \rangle$. We then perform an average over a thermal distribution of expectation values which we denote as $\langle \langle \hat{O} \rangle \rangle_{\text{th}}$. More precisely, we calculate thermal averages of a function, $A(\xi)$, of the continuous variable $\xi \equiv |\alpha_0|^2$ as

$$\langle A(\xi) \rangle_{\text{th}} \equiv \beta \int_0^\infty A(\xi)e^{-\beta \xi} d\xi, \tag{9}$$

where $\beta = \hbar \omega_0 / (k_B T)^{-1}$ for COM mode temperature $T$. For completeness, the Fock state calculations in the appendices involve the corresponding thermal average over discrete Fock state expectation values $A_n$ as

$$\langle A_n \rangle_{\text{th}} \equiv (1 - e^{-\beta}) \sum_{n=0}^{\infty} A_n e^{-\beta n}. \tag{10}$$

A. Bloch vector length ($\Delta \phi = 0$)

Here we are interested in calculating the expectation value of a component of the composite Bloch vector, $(\hat{S}_x, \hat{S}_y, \hat{S}_z) = (\sum_{i=1}^{N} \hat{S}_x^i, \sum_{i=1}^{N} \hat{S}_y^i, \sum_{i=1}^{N} \hat{S}_z^i)$, for initial spin states which are product states. This reduces to calculating the expectation value of a component of an individual spin $\hat{S}_i^z$. For the evolution operator of Eq. (6), which is a product of commuting displacement operators involving individual spins, only the displacement operator involving $\hat{S}_i^z$ nontrivially enters into the calculation. We assume that each experiment begins with the state $|\uparrow\rangle|\alpha_0\rangle = |\uparrow\rangle \hat{D}(\alpha_0)|0\rangle$, where $|\alpha_0\rangle$ is a coherent state of COM motion. In Secs. III A and III B, we consider a Ramsey pulse sequence consisting of two $\pi/2$ pulses separated by a time $\tau$ as shown in Fig. 3. For this Ramsey sequence consisting of a single ODF excitation period, Eq. (7) is used to calculate spin-dependent displacements, $\alpha$. The measurements of Sec. IV involve spin echo sequences, but all of the theory results of Secs. III A and III B apply with small modifications for calculating the spin-dependent displacement $\alpha$.

The first $\pi/2$ pulse of the sequence of Fig. 3 yields the qubit rotation

$$|\psi_1\rangle = \hat{R}(\pi/2, 0)|\uparrow\rangle|\alpha_0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \hat{D}(\alpha_0)|0\rangle, \tag{11}$$

(11)
Thermal averages may be performed over the continuous unity for the dephasing measurements discussed in Sec. IV. The quantum states \(|\psi_1\rangle\), \(|\psi_2\rangle\), and \(|\psi_f\rangle\) are labeled at appropriate points in the sequence as described in the text. For completeness, we present \(|\psi_f\rangle\), \(P_{1}\), and \(\langle \hat{S}_{z} \rangle\) resulting from the final \(\pi/2\) pulse, \(\hat{R}(\pi/2,0)\):

\[
|\psi_f\rangle = \frac{1}{2} \left[ (e^{-i\theta}|\alpha + \alpha_0\rangle - e^{i\theta}|\alpha - \alpha_0\rangle) + \frac{1}{2} \left( (e^{-i\theta}|\alpha + \alpha_0\rangle + e^{i\theta}|\alpha - \alpha_0\rangle) \right) \right].
\]

We now consider the effect of the spin-dependent ODF acting for the free evolution time \(\tau\). This yields the state

\[
|\psi_2\rangle = \frac{1}{\sqrt{2}} |\uparrow\rangle \hat{D}(\alpha) \hat{D}(\alpha_0)|0\rangle + \frac{1}{\sqrt{2}} |\downarrow\rangle \hat{D}(\alpha_0) \hat{D}(\alpha)|0\rangle
\]

\[
= \frac{1}{\sqrt{2}} e^{-i\theta_0}|\uparrow\rangle|\alpha + \alpha_0\rangle + \frac{1}{\sqrt{2}} e^{i\theta_0}|\downarrow\rangle|\alpha + \alpha_0\rangle,
\]

where \(\theta_0 = \text{Im}(\alpha|\alpha_0^\star|)^2\). It is useful to pause at Eq. (14) before applying the final microwave pulse and evaluate \(\langle \hat{S}_x \rangle\) and \(\langle \hat{S}_z \rangle\).

We find

\[
\langle \hat{S}_x \rangle = \frac{N}{2} \langle \hat{S}_z \rangle = \frac{N}{2} \cos(4 \text{Im}(\alpha^\star \alpha_0)) e^{-2|\alpha|^2},
\]

\[
\langle \hat{S}_z \rangle = \frac{N}{2} \langle \hat{S}_x \rangle = \frac{N}{2} \sin(4 \text{Im}(\alpha^\star \alpha_0)) e^{-2|\alpha|^2}.
\]

From Eqs. (15) and (16), we see that the ODF has caused a coherent rotation of the composite Bloch vector about the \(z\) axis by

\[
\theta_{\text{coh}} = \arctan \left( \frac{\langle \hat{S}_y \rangle}{\langle \hat{S}_z \rangle} \right) = 4 \text{Im}(\alpha^\star \alpha_0).
\]

The effect of spin-motion entanglement is reflected in the term \(e^{-2|\alpha|^2}\) of Eqs. (15) and (16), which deviates negligibly from unity for the dephasing measurements discussed in Sec. IV. Thermal averages may be performed over the continuous variable \(\xi \equiv |\alpha_0|^2\), where the magnitude \(\xi\) is now weighted according to Boltzmann statistics and the phase of \(\alpha_0 \in \mathbb{C}\) is evenly distributed over \(2\pi\) radians. Defining \(\alpha = |\alpha| e^{i\phi}\) and \(\alpha_0 = |\alpha_0| e^{i\phi_0}\), we calculate

\[
\langle \hat{S}_z \rangle_{\text{coh}} = \frac{\beta}{2\pi} \int_0^{2\pi} \int_0^\infty 16|\alpha|^2 \xi \sin^2(\phi' - \phi_0) e^{-\beta\xi} d\xi d\phi_0 = 8|\alpha|^2 \beta^{-1}.
\]

In Sec. IV, a typical \(|\alpha|\) for a 30 yN ODF driving at \(\omega_x\) for 100 \(\mu\)s is \(~ 0.05\) while \(\beta^{-1} \sim 12\) at the Be\(^{+}\) Doppler cooling limit, producing a nonnegligible rotation angle standard deviation of \(~ 30^\circ\).

We note that \(\beta^{-1} = \langle |\alpha_0|^2 \rangle_{\text{coh}}\). Also, for \(\bar{n} \gg 1\), \(\beta^{-1} \sim \bar{n}\) and the \(P_{4}\) of Eq. (21) agrees with the treatment that assumes a thermal distribution of Fock states [Eq. (A5)] given in Appendix A, but offers a more classical description. That is, for each possible coherent state amplitude and phase, the result of the experimental sequence is that the composite Bloch vector undergoes coherent rotation by some angle \(\theta_{\text{coh}}\). Over many such experimental sequences, we measure a dephasing of the composite Bloch vector associated with the angular variance, \(\langle \hat{S}_z^2 \rangle_{\text{coh}}\), of Eq. (18).

### B. Dephasing \((\Delta \phi = \pi/2)\)

A final qubit rotation of \(\hat{R}(\pi/2, \pi/2)\) \((\Delta \phi = \pi/2)\) transforms rotations and dephasing in the \(xy\) plane of the Bloch sphere to the detection \(z\) basis. Below we calculate the thermal average of the expectation value \(\langle \hat{S}_z^2 \rangle\) from which a temperature determination can be obtained. In addition, we discuss the implementation of a Monte Carlo analysis (see Sec. IV) that is in excellent agreement with measurements of thermal as well as nonthermal motional state distributions with \(\bar{n} \gg 1\).

For the \(\Delta \phi = \pi/2\) pulse sequence, we obtain the following expression for \(\langle \hat{S}_z \rangle\) after the final \(\pi/2\) pulse:

\[
\langle \hat{S}_z \rangle = \frac{N}{2} \sin(4 \text{Im}(\alpha^\star \alpha_0)) e^{-2|\alpha|^2}.
\]

Note that the thermal average of the expression in Eq. (24) vanishes, so we instead calculate dephasing through the second moment of \(\hat{S}_z\):

\[
\langle \hat{S}_z^2 \rangle = \frac{N}{4} \sum_{i=1}^{N} \langle \hat{S}_i^2 \hat{S}_i^2 \rangle + \frac{1}{4} \sum_{i \neq j} \langle \hat{S}_i^2 \hat{S}_j^2 \rangle
\]

\[
= \frac{N}{4} \frac{N(N-1)}{4} \langle \hat{S}_i^2 \hat{S}_j^2 \rangle,
\]

where \(\langle \hat{S}_i^2 \hat{S}_j^2 \rangle\) is an expectation value involving any two nonidentical spins within the ensemble. We simplify to
The collective spin of the trapped ions is first prepared in the state, $|\uparrow\rangle$, and the pulses at $\sim 124$ GHz constitute a Hahn spin echo that, in the absence of the ODF beams, leaves the spins in state $|\downarrow\rangle_N$ with $>99\%$ fidelity [34]. We apply the ODF laser beams during each free evolution period of the spin echo for a duration $\tau_1 = 500 \mu s$. The relative frequency of the two beams is $\mu$, and the relative phase of the ODF beat between the first and second arms is given by $\phi_{\text{ODF}}$. Note that for $\phi_{\text{ODF}} = \pi$ and $\mu = \omega_z$, the net result of the experimental sequence is identical to that of the Ramsey sequence of Fig. 3 with no intermediate $\pi$ pulse and a single free-precession period of $2\tau_1$. This is due to the condition that the ODF on state $|\uparrow\rangle$ ($F_1$) is opposite in sign but equal in magnitude to that on $|\downarrow\rangle$ ($F_1'$), where $(F_1 - F_1') = 2F_0$ [see Eq. (5)]. In this case the ODF of the second arm reinforces that of the first. In other words, the phase advance of the ODF beat by $\pi$ radians reverses the effect of the intermediate qubit $\pi$ pulse but retains the suppression of spin decoherence inherent in the spin echo. However, for the case $\mu \neq \omega_z$, the finite duration of the qubit $\pi$ pulse leads to a phase offset between the ion crystal oscillation and ODF drive at the start of the second arm given by $\delta (\tau_1 + t_0)$, where $\delta \equiv (\mu - \omega_z)$, that must be included in the theoretical analysis. Assuming that we interact exclusively with the COM mode, the final position $(\sigma_{\text{BE}})$ of the ion crystal in phase space after application of the two ODF pulses of Fig. 4(a) is given by

$$\sigma_{\text{BE}}(\tau) = \frac{F_{\text{ODF}}\omega_z}{2\hbar\sqrt{N}} \left(1 - e^{i\delta}\right)\left(\alpha_{\text{ODF}} - e^{i(\phi_{\text{ODF}} + \delta(t_1 + t_0))}\right),$$

where $\tau$ is the duration of each evolution period, $\phi_{\text{ODF}}$ is the ODF phase at the start of the first free evolution period, and we have included the additional $\phi_{\text{ODF}}$ to denote the added phase advance in the second free evolution period of the spin echo sequence. The common phase $\phi_{0}$ does not contribute to any experimental observables and may be disregarded. The relative minus sign between phase factors in the final term of Eq. (32) is due to the intermediate $\pi$ pulse of the spin echo sequence, which removes displacements common to both free evolution periods since $F_1 = -F_1'$. Using Eq. (32), we can estimate the phase-space excitation magnitude of lower frequency drumhead modes when resonantly exciting COM motion. The nearest mode to the COM is detuned by $\sim 10$ kHz [15]; therefore its maximum excitation magnitude relative to that of the COM mode is $\sim (2\tau_1)5^{-1}$, which gives more than an order-of-magnitude suppression for all data presented here.

Previous experiments with ions confined within RF Paul traps have shown that Doppler laser cooling produces thermal states of ion motion [35]. In Fig. 4(a), we show that measurements of the Bloch vector length under application of a spin-dependent force with $\mu \sim \omega_z$ are consistent with that of a thermal state of motion whose temperature is $0.4(1)$ mK, the Doppler cooling limit. The frequency width of the spectral feature is approximately given by the Fourier width of the ODF pulse duration of $2\tau_1 = 1$ ms, and the degree of decoherence measured near $\delta = 0$ is determined by the ODF magnitude and $\tilde{n}$ according to Eqs. (21) and (32) [15]. The detuning-independent background decoherence level of $P_c \sim 0.07$ in Fig. 4(a) is due to spontaneous emission from the off-resonant ODF laser beams, and is fully characterized for this system [15,36].
A. Thermal distributions

The measurement of Fig. 4(a) is one of composite Bloch vector length, and is only second-order sensitive to spin dephasing. To measure dephasing more directly and gain more complete knowledge of the spin statistics, we implement the experimental pulse sequence of Fig. 4(b). For this experiment, we resonantly drive the COM mode (δ = 0) with the spin-dependent ODF during free-evolution periods of τ2 = 100 μs. We set the phase of the final π/2 pulse of the sequence to be the same as the intermediate π pulse (Δφ = 3π/2), thereby rotating any dephasing within the xy plane to lie along the qubit axis. After each such sequence, we perform a projection measurement of Pτ and repeat for a total of 1000 experiments. The collection of 1000 Pτ values are binned and displayed as histograms in Fig. 4(b). As an additional check, we choose φODF to be either 0 or π. In the case of φODF = 0 (light gray bars), the spin echo effectively cancels the spin-dependent excitation and concomitant Bloch vector rotation, allowing for characterization of other sources of dephasing such as spin projection noise, photon shot noise, ac Stark shift fluctuations from the ODF lasers between the two spin echo arms, and excess magnetic field fluctuations not fully canceled by the spin echo. For φODF = π (red bars), the precession induced by the spin-dependent force in the first arm is enhanced in the second and we observe that the detected spin variance is greatly increased relative to the φODF = 0 case. We extract a COM temperature of 0.42(7) mK from this measurement by applying Eq. (31) to the excess variance of the φODF = π experiments. This temperature is in excellent agreement with the theoretical Doppler cooling limit as well as the Bloch vector length measurement of Fig. 4(a).

To further compare the measurements of Fig. 4(b) with the model of Sec. III B, we use a Monte Carlo algorithm to produce simulated histograms consisting of 50,000 “detection” events. This is a factor of 50 more detections than for the experimental measurements, and is so chosen to reduce noise in the Monte Carlo histograms for clearer distinction between experiment and simulation. Importantly, the simulations have no adjustable parameters—all inputs to the Monte Carlo algorithm (e.g., αSE, τ, τz) or obtained from measurement (e.g., ̄n from the measured σ). The simulated histograms are scaled vertically to match the experimental data given bin widths and total detection events.

The simulation procedure is the following: for each Monte Carlo run, k ∈ {1, . . . , 5 × 104}, we choose a random initial coherent state of COM motion given by |αk⟩eιφk. The probability of choosing a given magnitude, |αk|, is given by Boltzmann statistics for a thermal state. As such, the chosen magnitudes |αk| follow a probability distribution proportional to exp(−β|αk|2). The random value of φk is unweighted and assigned from the set [0,2π], which assumes the phase of the initial state of COM motion is uncorrelated with that of the ODF and varies for each experimental sequence. Following the pulse sequence of Fig. 4(b) with φODF = π, we apply relevant qubit rotations in sequence with coherent z-axis rotations due to the spin-dependent ODF given by Eqs. (15) and (16), where αk constitutes the initial coherent state for each run and αSE is the spin-dependent displacement calibrated as in Refs. [7,15].
For completeness, we also include the smaller measured variance in the \( \phi_{\text{ODF}} = 0 \) case phenomenologically by adding another, uncorrelated random rotation to the composite Bloch vector. The angle of this additional rotation follows a Gaussian probability distribution with the measured variance [light gray bars in Fig. 4(b)] \[37\]. The simulated histogram (black line) of Fig. 4(b) shows good agreement with experimental measurements, and further supports the absolute temperature measurement of Fig. 4(a). Note that the higher sensitivity of the direct spin-dephasing measurements enables the use of a shorter free-evolution period. As a result, direct measurements of spin dephasing are less sensitive to COM frequency and ODF phase drift within a single experiment. The effects of spontaneous emission decoherence are also negligible in this parameter regime.

**B. Thermal distributions with large coherent displacements**

We now describe the spin dephasing signature of relatively large coherent motional displacements, \( \alpha_d \), acting in addition to thermal fluctuations characterized by \( \bar{n} \). Such analysis may be relevant when narrow-bandwidth electric field fluctuations exist on trap electrodes at frequencies near that of the COM. As in the previous section, we assume that the phases of thermal, coherent, and ODF displacements are all uncorrelated. It may seem that little can be inferred from dephasing in the presence of coherent excitations with a random phase, but we demonstrate that a constant magnitude is all that is required to distinguish such noise sources.

We use resonant RF excitation of the COM mode to produce coherent states of motion whose square magnitude, \( |\alpha_d|^2 \), is larger than the thermal magnitude given by \( \bar{n} \). To this end, we apply an oscillating voltage with a frequency of \( \omega_z \) to the upper end cap electrode of the Penning trap for a period of 20 \( \mu \)s as in Ref. \[38\]. This homogeneous COM mode excitation is applied following the initial Doppler cooling and state preparation pulses but before the experimental pulse sequence of Fig. 4(b). The amplitude of the applied voltage is 110 \( \mu \)V (1.2 mV m\(^{-1}\) at the ion position), corresponding to a coherent excitation of \( |\alpha_d| \sim 8 \) (\( \sim 30 \) nm amplitude). Histograms compiled from 1000 experimental runs without and with the resonant RF excitation are shown in Figs. 5(a) and 5(b), respectively. The absolute COM temperature is determined to be 0.7(1) mK for the sequence without coherent excitation. This elevated COM temperature is due to the connection of the direct digital synthesizer used to apply resonant RF to the upper end cap. Nevertheless, we achieve \( |\alpha_d|^2/\bar{n} \sim 3.5 \) for these experiments. Figure 5(b) shows the effect of applying the coherent motional excitation, namely that the spin distribution is split into two peaks whose separation is determined by the combined coherent and ODF displacements. If the relative phase of \( \alpha_d \) and \( \alpha_{SE} \) are fixed for every experimental sequence, then the mean of the histogram of Fig. 5(a) would simply be shifted to a new position corresponding to a coherent Bloch vector rotation, with the standard deviation still reflecting the COM temperature of 0.7(1) mK. However, the randomness of the relative phases of \( \alpha_d \) and \( \alpha_{SE} \) in combination with their constant amplitudes leads to a characteristic splitting of the spin distribution about the mean.

![Figure 5](image-url)

**C. Heating rates**

Motional heating in RF ion traps has gained increased prominence in recent years due primarily to studies of so-called “anomalous heating” from ion-surface proximity \[39–44\]. Recent work suggests surface contamination as the culprit \[45,46\]. Additionally, micromotion in RF traps limits the usable size of Coulomb crystals for quantum information and quantum simulation experiments. Penning traps use static...
collisions are a primary contributor [49]. Heating rates of 

crystals in our Penning trap estimate that background gas

of global (not mode-resolved) ambient heating of large 3D

modes of an ion crystal in a Penning trap. A previous study

yet been reported for ambient heating of a resolved motional

means a likely insensitivity to anomalous heating processes.

Furthermore, the large physical size of typical Penning traps

potentials for ion confinement and therefore do not induce micro-

motion, enabling the formation of large ion crystals [47,48].

Furthermore, the large physical size of typical Penning traps

means a likely insensitivity to anomalous heating processes.

Despite these encouraging features, no measurements have

yet been reported for ambient heating of a resolved motional

mode of an ion crystal in a Penning trap. A previous study of

global (not mode-resolved) ambient heating of large 3D
crystals in our Penning trap estimates that background gas

collisions are a primary contributor [49]. Heating rates of

∼65 mK/s were measured for background pressures of ∼

4 × 10^{-9} Pa (3 × 10^{-11} Torr), which translates to

dt\sim 1.7 \times 10^3 s^{-1} at our trap frequency of 795 kHz.

We apply the thermometry techniques summarized in Fig. 4

to obtain an initial measurement of the ambient heating rate of

the axial COM mode of our 2D crystals. We include a variable

delay between initial Doppler cooling/state preparation and

application of the experimental pulse sequences. Any increase

in \bar{n} over this initial delay period is measured in the subsequent
decoherence or dephasing measurement. Figure 6 shows

measured absolute COM temperatures as a function of initial
delay (points with error bars) along with linear fits to each
data set (solid lines). The data of Figs. 6(a) and 6(b) are

identical, but plotted on linear and logarithmic vertical axes,

respectively, for clarity. The fitted slopes reflecting

\frac{dt}{\bar{n}} for each curve are displayed in Fig. 6(b) and color coded to match the

corresponding data set.

We measure the largest heating rate of 1.4(2) \times 10^4 s^{-1}
(black points) when the trap end cap electrodes are held at
0 V using the high-voltage power supplies responsible for

initial ion loading and transport. Small voltage fluctuations
from these power supplies as well as electromagnetic
interference along the connecting cables are the likely cause
of this COM heating. Upon grounding the end caps directly
to the vacuum chamber at the high-voltage feedthrough, we

observe an order-of-magnitude drop in the heating rate to

between 4.7(8) \times 10^2 s^{-1} (blue points) and 1.2(3) \times 10^3 s^{-1}
(red points). This approximate factor-of-two variation in

heating rates is representative of our day-to-day observations

with different ion samples, and we see no evidence of a
correlation between total ion number and heating rate with the

end cap electrodes grounded at the feedthrough. The mean of

the two lowest heating rates corresponds to ∼30 mK/s, which

is near previous collisional heating estimates [49]. Additional

filtering of the central ring and rotating-wall electrodes outside

the vacuum envelope yielded no measurable improvement in

COM heating [50].

Because the distance from the trapped-ion arrays to the

trap electrode surfaces (≥2 cm) is large compared to the

diameter of the planar array (<0.5 mm), electric field noise

from trap electrode surfaces will be uniform across the array

and preferentially heat the COM mode. For an array with N

ions this results in a linear dependence of the COM heating rate

ion number due to uniform electric field noise [4]. The lack of

an observed N dependence in the measured ambient-heating

rate indicates the source of the heating is likely not electric field

noise. However, any potential anomalous heating must be less

than the measured ∼10^3 s^{-1} ambient-heating rate. Dividing

this limit by the number of trapped ions (N ∼ 200) gives a

limit on the anomalous heating rate for a single trapped ion of

∼5 s^{-1} at a trap frequency of 795 kHz.

V. CONCLUSION

In summary, we have demonstrated a technique for an-
alyzing the motional state of a resolved ion crystal mode.
The methods presented here do not rely on stimulated
Raman transitions or Doppler linewidth analysis, and are in
principle applicable to any resolved motional mode at any
temperature provided the Lamb-Dicke confinement criterion
is satisfied for the given mode. The sensitivity of our spin

dephasing measurements allows for a regime of operation

with negligible spin-motion entanglement and spontaneous
emission decoherence. Monte Carlo simulations based on the

semiclassical description of Sec. III are in excellent agreement

with spin dephasing measurements. Furthermore, we observe

a clear distinction between coherent and thermal states of

motion, despite the randomness of the RF drive phase relative
to our optical dipole force.

The methods and analysis presented here enable very

sensitive detection of coherently driven motion of a trapped-

ion crystal, and may be used to phase-sensitively detect weak

forces [38,51]. Section III can be used to estimate and optimize

the force detection sensitivity for a given N and temperature

of the trapped-ion crystal. We estimate that the spin-motion
coupling technique discussed here could improve on the force

sensitivity obtained in Ref. [38] by more than an order of

magnitude.

We also present measurements of ambient heating of a re-
solved mode of motion in a Penning trap. Future crystal heating

measurements will include other resolved transverse motional

modes with the goal of more clearly distinguishing between

electric field fluctuations (mode-specific, N-dependent) and

background gas collisions (mode- and N-independent). Nev-

ertheless, we demonstrate that low heating rates are indeed

achievable in Penning ion traps.

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APPENDIX A: CALCULATING BLOCH VECTOR LENGTH USING FOCK STATES

A thermal motional state is described by a density matrix that is a statistical mixture of Fock states. This motivates a calculation that assumes each experiment begins with the system in the state, \(|\uparrow\rangle|n\rangle\), where \(|n\rangle\) is the harmonic oscillator Fock state of COM motion. The first \(\pi/2\) pulse of the Ramsey sequence yields the qubit rotation

\[
|\psi_1\rangle = \hat{R}(\frac{\pi}{2},0)|\uparrow\rangle|n\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle + |\downarrow\rangle)|n\rangle. \tag{A1}
\]

The spin-dependent ODF then produces displaced Fock states, \(|\alpha,n\rangle\), as \([52]\)

\[
|\psi_2\rangle = \hat{D}_{SD}(\alpha)|\psi_1\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle|\alpha,n\rangle + |\downarrow\rangle| - \alpha,n\rangle). \tag{A2}
\]

Note that \(|\psi_2\rangle\) involves entanglement of spin and motional degrees of freedom for nonzero \(\alpha\). We now apply the final \(\pi/2\) pulse whose phase is identical to the first \((\Delta \phi = 0)\) to obtain

\[
|\psi_f\rangle = \frac{1}{2}|\uparrow\rangle (|\alpha,n\rangle - | - \alpha,n\rangle) + \frac{1}{2}|\downarrow\rangle (|\alpha,n\rangle + | - \alpha,n\rangle). \tag{A3}
\]

The probability of measuring \(|\uparrow\rangle\) for state \(|\psi_f\rangle\) depends on the overlap of \(|\alpha,n\rangle\) and \(| - \alpha,n\rangle\), and is given by

\[
P_\uparrow = \frac{1}{2} [1 - L_n(4|\alpha|^2)e^{-2|\alpha|^2}]. \tag{A4}
\]

where \(L_n\) is the Laguerre polynomial of order \(n\). Our fluorescence detection is insensitive to the ion motional state, so we perform a Boltzmann-weighted thermal average over all Fock states to obtain

\[
P_\uparrow \equiv \langle P_\uparrow^{(n)}\rangle_{th} = \frac{1}{2} \left[1 - e^{-2|\alpha|^2/\beta} \right]. \tag{A5}
\]

In the above equation, \(\beta = (e^\beta - 1)^{-1}\) is the average COM mode occupation number for a thermal state at temperature \(T\), and \(\beta = \hbar\omega / k_B T\). In the absence of the spin-dependent displacement, \(P_\uparrow = 0\) for this pulse sequence. However, as the displacement amplitude increases, \(P_\uparrow\) takes on positive values that increase with mode occupation—eventually saturating at \(P_\uparrow = 0.5\) corresponding to complete loss of spin coherence. In Ref. [15], we describe using this decoherence signature to perform mode spectroscopy and thermometry on a planar array of ions.

We may also calculate the expectation value for the \(N\)-spin composite Bloch vector, \(\langle \hat{S}_z \rangle = \sum_{i=1}^{N} \langle \hat{S}_z^i \rangle = \frac{N}{2} \langle \hat{S}_z \rangle\) using the final state of Eq. (A3). We obtain

\[
\langle \hat{S}_z \rangle = -\frac{N}{2} L_n(4|\alpha|^2)e^{-2|\alpha|^2}, \tag{A6}
\]

\[
\langle \langle \hat{S}_z \rangle \rangle_{th} = -\frac{N}{2} e^{-2|\alpha|^2(2\beta+1)}. \tag{A7}
\]

We note that the calculation in this appendix, which uses the Fock state basis, motivates a picture of spin decoherence produced by entanglement of the spin and motional degrees of freedom. However, in the manuscript we show that for coherent input states of motion, spin decoherence can be explained by dephasing without resorting to quantum entanglement of spin and motion.

APPENDIX B: CALCULATING DEPHASING USING FOCK STATES

If the phase of the final microwave \(\pi/2\) pulse is shifted by \(\pi/2\) [e.g., \(\hat{R}(\frac{\pi}{2},\frac{\pi}{2})\), \(\Delta \phi = \pi/2\)], then the composite Bloch vector will remain in the equatorial plane of the Bloch sphere and \(\langle \hat{S}_z \rangle = \langle \hat{S}_z \rangle_{th} = 0\). Fock states therefore produce no coherent spin rotation due to the spin-dependent ODF.

To calculate dephasing, we must compute pairwise spin correlations of the form \(\langle \hat{S}_z^1\hat{S}_z^2 \rangle\) as shown in Eq. \((26)\). As in Appendix A, we will consider the initial state of COM motion to be a Fock state, \(|n\rangle\). In contrast with the Bloch vector length calculation that requires only a single spin, we here consider a two-spin system whose full initial state is \(|\uparrow\rangle|n\rangle\). We construct the necessary two-spin rotation matrices, \(\hat{R}^{(2)}(\frac{\pi}{2},0)\) and \(\hat{R}^{(2)}(\frac{\pi}{2},\frac{\pi}{2})\), using Kronecker products as follows:

\[
\hat{R}^{(2)}(\frac{\pi}{2},0) = \hat{R}(\frac{\pi}{2},0) \otimes \hat{R}(\frac{\pi}{2},0) = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix}, \tag{B1}
\]

\[
\hat{R}^{(2)}(\frac{\pi}{2},\frac{\pi}{2}) = \hat{R}(\frac{\pi}{2},\frac{\pi}{2}) \otimes \hat{R}(\frac{\pi}{2},\frac{\pi}{2}) = \frac{1}{2} \begin{pmatrix} 1 & i & i & -1 \\ i & 1 & 1 & i \\ 1 & -1 & 1 & 1 \\ -i & -1 & -1 & i \end{pmatrix}. \tag{B2}
\]

We can now define the final state, \(|\psi_f\rangle\), after the pulse sequence of Fig. 3 \((\Delta \phi = \frac{\pi}{2})\) as

\[
|\psi_f\rangle = \hat{R}^{(2)}(\frac{\pi}{2},\frac{\pi}{2}) \hat{D}_{SD}(\alpha) \hat{R}^{(2)}(\frac{\pi}{2},0)|\uparrow\rangle|n\rangle, \tag{B3}
\]

where \(\hat{D}_{SD} = \exp[(\alpha \hat{a}^\dagger - \alpha^* \hat{a}) \sum_{i=1}^{N} \hat{S}_z^i] \) and \(\alpha\) is the spin-dependent displacement amplitude. The expectation value \(\langle \hat{S}_z^1\hat{S}_z^2 \rangle = \langle \psi_f| \hat{S}_z^1\hat{S}_z^2|\psi_f\rangle\) and corresponding thermal average are

\[
\langle \hat{S}_z^1\hat{S}_z^2 \rangle = \frac{1}{2} \left[1 - L_n(16|\alpha|^2)e^{-8|\alpha|^2} \right], \tag{B4}
\]

\[
\langle \langle \hat{S}_z^1\hat{S}_z^2 \rangle \rangle_{th} = \frac{1}{2} \left[1 - e^{-8|\alpha|^2(2\beta+1)} \right]. \tag{B5}
\]

As is the case with the Bloch vector length calculation, Eq. \((B5)\) agrees with our earlier result using coherent states \([Eq. (30)]\) in the limit \(\beta \rightarrow 1\).

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[25] This background dephasing contributes negligibly to the final dephasing of interest for the given experimental conditions, but is described here for completeness.